MODULAR SYSTEM

LOGARITHMS

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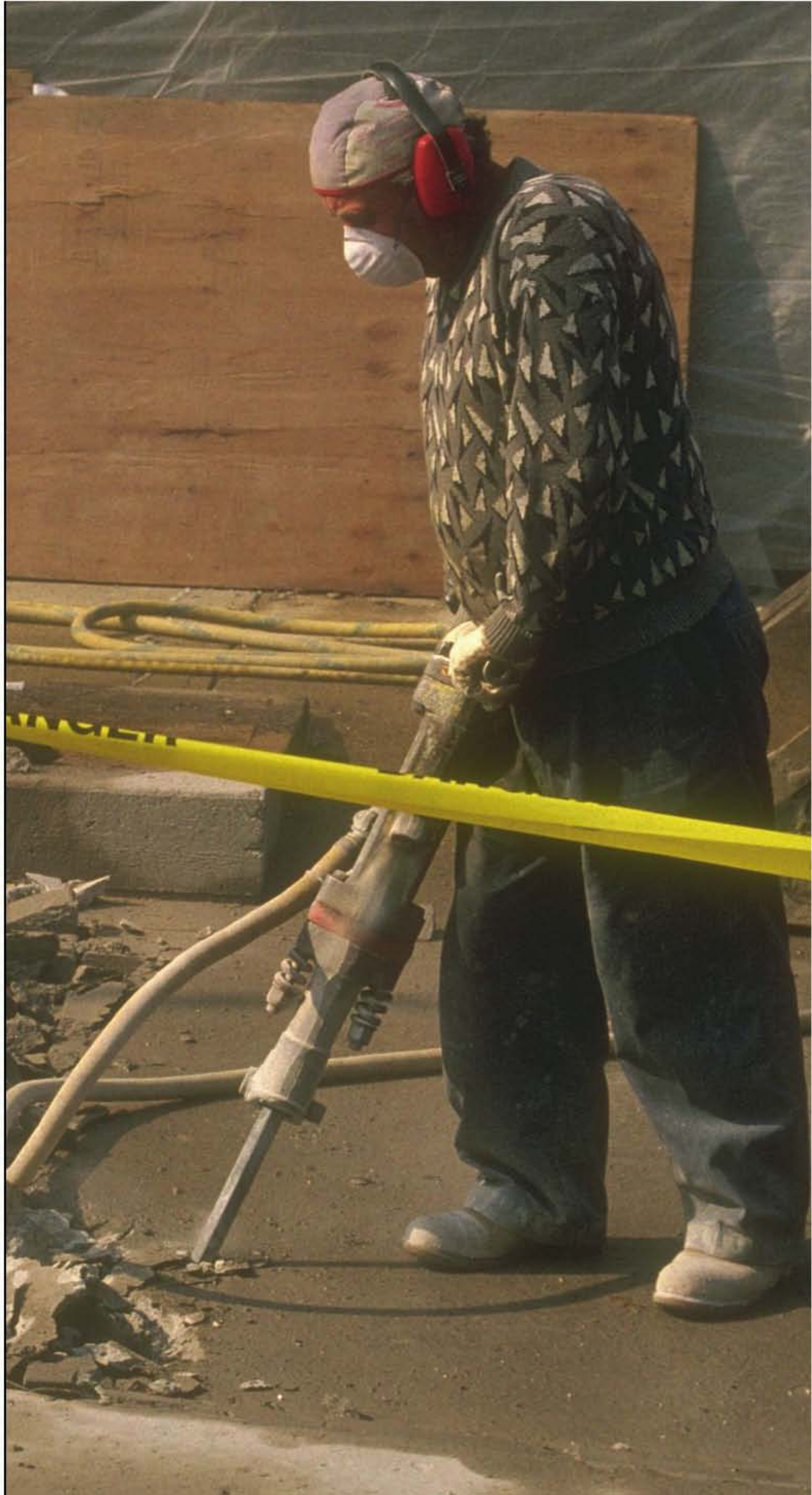


PREFACE

To the Teacher

This book constitutes a high-school course in exponential and logarithmic functions. Since the concepts of logarithm and logarithmic function are new to most students at this level, the material assumes no prior knowledge of the subject, although students should be familiar with real numbers and their properties, with exponents and their properties, and with linear and quadratic functions and their graphs. In particular, students should have studied the Numbers and Functions modules in the Zambak Modular System mathematics series before beginning this book.

The book is divided into three chapters, structured as follows:

- *Chapter 1 deals with exponents and exponential functions. It begins with a review of the basic properties of exponents, followed by an introduction to exponential functions. An optional third section discusses the graphs of exponential functions and some simple variations, and the fourth section looks at applications of exponential functions.*

  Exponents and Exponential Functions

- *Chapter 2 introduces logarithms and logarithmic functions and their properties. The third section (again optional) looks at the graphs of logarithmic functions and some standard transformations. The fourth section covers three applications of logarithmic functions in the real world, namely the Richter scale, the pH scale and the decibel scale.*

  Logarithms and Logarithmic Functions

- *Chapter 3 looks at equations and inequalities, building on the material covered in the first two chapters. The text presents exponential equations and inequalities in the first section, and moves on to logarithmic equations and inequalities in the second. The third and final section looks at systems of equations and inequalities.*

  Exponential and Logarithmic Equations and Inequalities

This book has been designed to be an effective teaching aid and includes all of the features of the Zambak Modular System mathematics series, as described in the section Using This Book on the next page.

Acknowledgements

This module could not have been written without a great deal of help from many people. I would like to express my thanks to everyone who took time to review the material and make invaluable suggestions. In particular, I would like to thank Mustafa Kırıkçı and Ali Çakmak at Zambak Publishing for their encouragement and patience during the writing of this module. I would also like to thank Ramazan Şahin, who initially instigated the project and who gave me so much good advice at the beginning of my authoring days. Following on from him, Cem Giray and Ali Lafçıoğlu made invaluable contributions to the module, for which I am very grateful.

Like any author, I remain indebted to all those who do their best to improve on my best. I am particularly grateful to the following people who all helped to take my work from its first draft to its final form:

- *to the authors of other titles in the Modular System series, for their many helpful comments, criticisms and suggestions;*

- *to the administration of my place of work, for their support during the writing of the book and for providing the necessary facilities that I too often take for granted;*

– to the staff of the design center at Zambak Publishing, especially Serdar Çam, Şamil Keşkinoğlu and Serdal Yıldırım, for their patient typesetting and design;

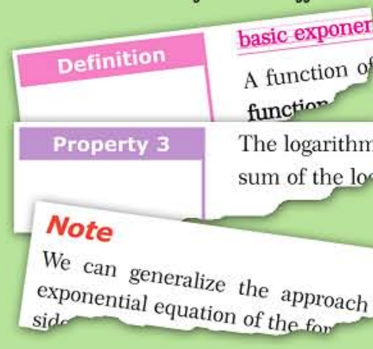
– to Zoe Barnett, for her careful proofreading of the main text.

Most importantly I thank my wife and children, who put up with a string of trips abroad, lost weekends and odd working hours during the project.

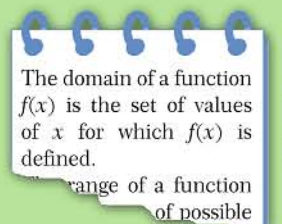
Salih Katırcı

To the Student: Using This Book

This book is designed so that you can use it effectively. Different pieces of information are useful in different ways. Look at the types of information, and how they appear in the book:

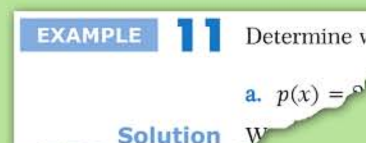


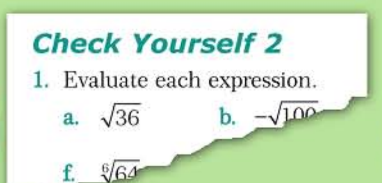
Definition boxes give formal definitions of new concepts. Property boxes present properties that can often be proved. Notes help you focus on important details. All of these things help you to understand the text and the examples.



A small notebook in the left or right margin of a page reminds you of the math you need so that you can understand the material.

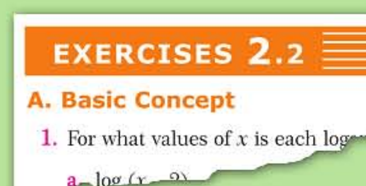
Examples show problems related to the topic and their solution, with explanations. The examples are numbered, so that you can find them easily in the book.

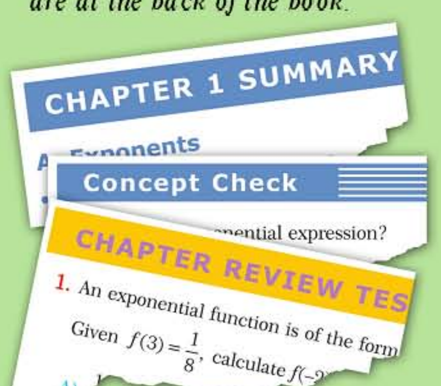




Questions in Check Yourself sections help you check your understanding of what you have just studied. Solve these questions alone and then compare your answers with the answer key provided.

Exercises at the end of each section cover the material in the whole section. You should be able to solve all the problems which do not have a star. Questions marked (✪) and (✪✪) are more difficult. The answers to the exercises are at the back of the book.





The Chapter Summary at the end of each chapter summarizes all the important material that has been covered in the chapter. The Concept Check section contains oral questions that you should be able to answer from reading the text, or by exploring the topic in other books or on the Internet. Chapter Review Tests contain multiple-choice questions to help you prepare for exams. The answer key for these tests is at the back of the book.

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LOGARITHMS

CHAPTER 1: EXPONENTS AND EXPONENTIAL FUNCTIONS

CHAPTER 2: LOGARITHMS AND LOGARITHMIC FUNCTIONS

CHAPTER 3: EXPONENTIAL AND LOGARITHMIC EQUATIONS AND INEQUALITIES

INTRODUCTION

"... by shortening the labours doubled the life of the astronomer."

Pierre Simon Laplace
about Napier's logarithms

Today, many people only think of logarithms as an unnecessary and difficult part of mathematics. However, logarithms were (and still are) a very useful calculating tool.

It is easy to overlook logarithms in our modern world, when all we have to do is enter numbers into a calculator or computer to get solutions to problems that might have taken hours, days or years to solve before the invention of logarithms. Logarithms speed up multiplication and division calculations by converting the operations to addition and subtraction. The only additional work added to the process is looking up logarithms and antilogarithms in tables.

John Napier

Although there is evidence that logarithms were known in 8th century India, their invention as an aid to calculation is attributed to John Napier (1550-1617). Napier lived at a time of great new developments in the world of astronomy. Many astronomers were calculating and re-calculating the positions of the planets using Copernicus's theory of the solar system, which was published in 1543. Their calculations took up pages and pages and hours and hours of work. While working on his famous laws of planetary motion, Johannes Kepler (1571-1630) still had to fill nearly 1000 large pages with calculations! But by using logarithms, Kepler was able to reduce his working and make his breakthrough.

Copernicus

Johannes Kepler

Napier released his first logarithmic tables in 1614 in his paper *Mirifici logarithmorum canonis descriptio*, using the series $10^7(1 - 10^{-7})^n$, $n = 0, 1, 2, \dots, 100$.

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SIVE
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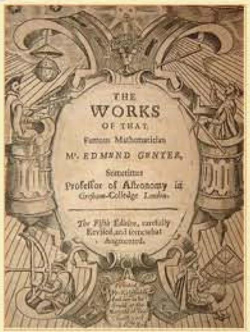
NVLLA PRÆSTITVTA DIE.
LONDINI,
Excudebat GVLIELMVS
IONES. 1624.

Napier did not talk about a base in his tables (actually the base was $1/e$), wrote the logarithm of x. 'As Nap.log x'. To describe his invention, he used the Latin word *logarithmus*, which derives its meaning from two Greek words: *logos*, which means a principle relationship between numbers or ratio, and *arithmos*, meaning 'number'.

Meanwhile, the British mathematician Henry Briggs (1561-1630) realized the importance of Napier's work and moved to Scotland to meet Napier. Briggs modified Napier's original idea but used a simple geometrical series to calculate logarithms, based on powers of 10. In 1617, Briggs published his first table of these logarithms to eight decimal places. Briggs later wrote *Arithmetica logarithmica* (1624), a work which contained logarithmic tables for 30,000 natural numbers to 14 decimal places. His logarithms are known today as common logarithms.

Other tables followed which helped to spread the use of logarithms, especially in Europe. The Swiss clockmaker and mathematician Joost Bürgi had worked with Kepler on different problems. Bürgi had also been working independently on logarithms at the same time as Napier. He had a slightly different approach to Napier's which used the indices of geometric progressions. Burgi's work clearly shows the operation and nature of logs, but it was only published in 1620 and is basically a set of anti-logarithm tables.

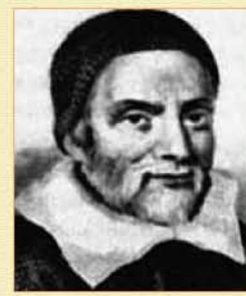
Jost Bürgi



The discovery of logarithms led to the invention of an important calculating device: the slide rule.

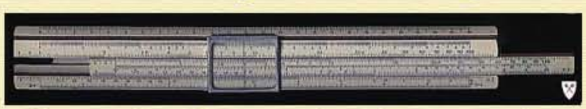
Edmund Gunter (1581-1626) was the first to propose a 'calculating line' of logarithms in 1624. Gunter's invention (later called 'Gunter's Line of Numbers') used scales divided into line segments. The length of each segment was proportional to the logarithm of a number. By measuring line segments with a compass, mathematicians could multiply and divide numbers easily. Gunter's Line of Numbers quickly became a popular calculating tool.

In 1630, William Oughtred (1574-1660) suggested printing the same scale on two rules which could slide one next to the other. This made the use of a compass unnecessary, and gave the slide rule its current form.



William Oughtred

Ismail Gelenbevi (1730-1790) was an ottoman mathematician and academic. Gelenbevi published around thirty-five scientific articles written in Turkish and Arabic, and is credited with the introduction of logarithms to Turkey. Gelenbevi's *Risala fi Sharh-i Jadavil al Ensab* concerns tables of logarithms and their use, a topic which was becoming known in Istanbul at the time. This treatise was the first independent work on logarithms in the Ottoman era.



Today, slide rules have been replaced by scientific calculators and computers. However, a good understanding of logarithms and antilogarithms (exponents) and the reasoning behind them will help to develop your mathematical thinking skills in many areas. In addition, logarithms and exponential functions are useful in other subjects including chemistry, physics, economics and the engineering sciences. And finally, logarithms have many practical uses. They give us a short-hand, concise way of working with data that covers a very large range of values. A variety of things such as the acidity of a solution, noise levels, and the severity of an earthquake are conventionally measured using scales based on logarithms.

Chapter 1

EXPONENTS AND EXPONENTIAL FUNCTIONS

1 EXPONENTS

Up to now your study of math has included the study of linear, quadratic and trigonometric functions. In this book we will look at two new types of function: exponential functions and logarithmic functions. As we shall see, these functions are inverses of each other.

Let us begin by recalling what you already know about exponents and exponential expressions. In the exponential expression a^p, a is called the base and p is called the exponent. We read this expression as 'the pth power of a', or 'p to the power a'. We can also read a^2 as 'a squared' and a^3 as 'a cubed'.

A. INTEGER EXPONENTS

An exponential expression is a short-hand way of writing a multiplication operation in which the factors are all equal. In other words, for a natural number n we can write

$\boxed{a^n = a \cdot a \cdot a \cdot \ldots \cdot a}$ where $a \in \mathbb{R}$ appears as a factor n times.

Remember that the first power of any number is the number itself, and any non-zero number to the power zero has value 1:

$\boxed{a^1 = a,\ a^0 = 1 \text{ where } a \neq 0, \text{ and } 0^0 \text{ is undefined.}}$

We can also define the negative power of a non-zero base as

$\boxed{a^{-n} = \dfrac{1}{a^n}}$ where $a \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$.

Finally, remember that in an expression such as $2a^5$, 5 is the exponent of a, not the exponent of $2a$. Similarly, -2^4 and $(-2)^4$ are different: $-2^4 = -16$ while $(-2)^4 = 16$.

EXAMPLE 1 Evaluate the expressions.

a. 5^0 b. $(-7)^0$ c. 4^1 d. 2^{-2}

Solution We can use the properties given above.

a. $5^0 = 1$ b. $(-7)^0 = 1$ c. $4^1 = 4$ d. $2^{-2} = \dfrac{1}{2^2} = \dfrac{1}{4}$

Using the properties above, we can easily derive the following additional properties for any integers m and n and any non-zero real numbers a and b:

1. $a^m \cdot a^n = a^{m+n}$	2. $\dfrac{a^m}{a^n} = a^{m-n}$	3. $(a^m)^n = a^{m \cdot n}$
4. $(a \cdot b)^m = a^m \cdot b^m$	5. $(\dfrac{a}{b})^m = \dfrac{a^m}{b^m}$.	

EXAMPLE 2 Simplify each expression, given that all variables represent positive real numbers. Leave each answer as a single number or fraction with positive exponents.

a. $7^6 \cdot 7^4$ b. $\dfrac{9^6}{9^3}$ c. $(2x)^3$ d. $\dfrac{x^{-2}-y^{-2}}{x^{-1}-y^{-1}}$

Solution a. $7^6 \cdot 7^4 = 7^{6+4} = 7^{10}$ b. $\dfrac{9^6}{9^3} = 9^{6-3} = 9^3$ c. $(2x)^3 = 2^3x^3 = 8x^3$

d. $$\frac{x^{-2}-y^{-2}}{x^{-1}-y^{-1}} = \frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}-\frac{1}{y}} = \frac{\frac{y^2-x^2}{x^2\cdot y^2}}{\frac{y-x}{x\cdot y}} = \frac{y^2-x^2}{x^2\cdot y^2}\cdot\frac{x\cdot y}{y-x}$$

$$= \frac{(\cancel{y-x})(y+x)}{(x\cdot y)^{\cancel{2}}}\cdot\frac{\cancel{x\cdot y}}{\cancel{y-x}}$$

$$= \frac{x+y}{x\cdot y}$$

Be careful!

$7^6 \cdot 7^4 \neq 7^{24}$

$\dfrac{9^6}{9^3} \neq 9^2$

$(2x)^3 \neq 2x^3$

$a^2 - b^2 = (a-b)\cdot(a+b)$

Check Yourself 1

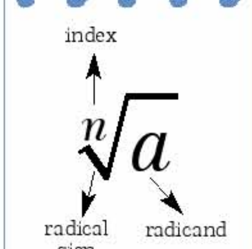
Simplify each expression, given that all variables are positive. Leave each answer as a single number or fraction with positive exponents.

a. $3^3 \cdot 3^4$ b. $\dfrac{7^8}{7^2}$ c. $3(x^2y)^2$ d. $\dfrac{x^{-1}+y^{-1}}{x^{-2}-y^{-2}}$

Answers

a. 3^7 b. 7^6 c. $3x^4y^2$ d. $\dfrac{xy}{y-x}$

B. ROOTS AND RADICAL EXPRESSIONS



Let a and b be real numbers, and let n be an integer greater than or equal to 2. Then an nth root of a is a number which, when raised to the power n, is equal to a. In other words, b is an nth root of a if and only if $b^n = a$.

For example, 4 is a third root of 64 because $4^3 = 64$, and –2 is a third root of –8 because $(-2)^3 = -8$. Moreover, both 3 and –3 are second roots of 9 because $3^2 = (-3)^2 = 9$. Note that a second root is usually called a square root and a third root is called a cube root.

We write $\sqrt[n]{a}$ to mean the principal nth root of a. In this notation, a is the radicand, n is the index, and the symbol $\sqrt{}$ is called the radical sign. We do not write the index when $n = 2$.

The principal nth root $\sqrt[n]{a}$ identifies

a. the positive root of a when n is even and a is positive.

b. the unique root which has the same sign as a when n is odd.

When n is even and a is negative, the principal nth root of a is undefined in the set of real numbers.

By the definition above we can write $\sqrt[3]{64}=4$, $\sqrt[3]{-8}=-2$, $\sqrt{9}=3$ and $\sqrt[5]{32}=2$.

We can also write parts a and b of the definition above as follows:

$$\text{For } n\geq 2,\ n\in\mathbb{N}, \text{ and } a\in\mathbb{R},\quad \sqrt[n]{a^n}=\begin{cases} a & \text{if } n \text{ is odd.}\\ |a| & \text{if } n \text{ is even.}\end{cases}$$

Any expression in the form $\sqrt[n]{a}$ is called a radical expression, or simply a radical. We can write radical expressions in different ways:

$$\sqrt[3]{64}=\sqrt[3]{4^3}=4,\quad \sqrt[3]{-8}=\sqrt[3]{(-2)^3}=-2,\quad \sqrt{9}=\sqrt{(-3)^2}=\sqrt{3^2}=3 \text{ and } \sqrt[5]{32}=\sqrt[5]{2^5}=2.$$

EXAMPLE 3 Evaluate each radical expression.

a. $\sqrt[3]{125}$ b. $\sqrt[3]{-64}$ c. $\sqrt{(-5)^2}$ d. $\sqrt[4]{-16}$

Solution a. $\sqrt[3]{125}=\sqrt[3]{5^3}=5$ b. $\sqrt[3]{-64}=\sqrt[3]{(-4)^3}=-4$ c. $\sqrt{(-5)^2}=|-5|=5$

d. Since the radicand (-16) is negative and the index is even, the radical $\sqrt[4]{-16}$ is undefined in the set of real numbers.

We can use the following laws to evaluate and simplify radical expressions:

$$1.\ \sqrt[n]{a\cdot b}=\sqrt[n]{a}\cdot\sqrt[n]{b}\qquad 2.\ \sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}\qquad 3.\ \sqrt[n]{\sqrt[m]{a}}=\sqrt[m\cdot n]{a}.$$

EXAMPLE 4 Simplify the expressions.

a. $\sqrt[3]{\sqrt{11}}$ b. $\sqrt[3]{\frac{8}{27}}$ c. $\sqrt[5]{8a^2}\cdot\sqrt[5]{4a^3}$

Solution a. $\sqrt[3]{\sqrt{11}}=\sqrt[3\cdot 2]{11}=\sqrt[6]{11}$ b. $\sqrt[3]{\frac{8}{27}}=\frac{\sqrt[3]{8}}{\sqrt[3]{27}}=\frac{\sqrt[3]{2^3}}{\sqrt[3]{3^3}}=\frac{2}{3}$

c. $\sqrt[5]{8a^2}\cdot\sqrt[5]{4a^3}=\sqrt[5]{(8a^2)(4a^3)}=\sqrt[5]{32a^5}=\sqrt[5]{2^5a^5}=\sqrt[5]{(2a)^5}=2a$

Check Yourself 2

1. Evaluate each expression.

a. $\sqrt{36}$ b. $-\sqrt{100}$ c. $\sqrt[3]{729}$ d. $\sqrt[3]{-343}$ e. $\sqrt[5]{-1}$

f. $\sqrt[6]{64}$ g. $\sqrt[6]{-729}$ h. $\sqrt{\frac{25}{49}}$ i. $\sqrt[3]{-\frac{8}{125}}$ j. $\sqrt[3]{\frac{343}{1000}}$

Answers

1. a. 6 b. –10 c. 9 d. –7 e –1

f. 2 g. undefined in $\mathbb{R}$ h. $\frac{5}{7}$ i. $-\frac{2}{5}$ j. $\frac{7}{10}$

C. RATIONAL EXPONENTS

$a^m \cdot a^n = a^{m+n}$

$a^m \cdot b^m = (a \cdot b)^m$

An exponent of the form $\frac{1}{n}$ where $n \in \mathbb{N}$ and $n \geq 2$ indicates a root with index n: $\boxed{a^{\frac{1}{n}} = \sqrt[n]{a}}$.

For example, $3^{\frac{1}{2}} = \sqrt{3}$ and $5^{\frac{1}{3}} = \sqrt[3]{5}$.

The expression $a^{\frac{1}{n}}$ is equivalent to the principal nth root of a, and so it satisfies the properties of principal roots. For example, $(-4)^{\frac{1}{2}}$ is undefined in the set of real numbers because $(-4)^{\frac{1}{2}} = \sqrt{-4}$, which is not a real number.

Taking into account the definition of the nth root of a number we can also write

$$\boxed{a^{\frac{1}{n}} = b \text{ if and only if } b^n = a.}$$

EXAMPLE 5 Evaluate each expression.

a. $100^{\frac{1}{2}}$ b. $81^{\frac{1}{4}}$ c. $64^{\frac{1}{6}}$ d. $(-32)^{\frac{1}{4}}$ e. $(-32)^{\frac{1}{5}}$ f. $(32)^{-\frac{1}{5}}$

Solution a. $100^{\frac{1}{2}} = \sqrt{100} = 10$

b. $81^{\frac{1}{4}} = \sqrt[4]{81} = 3$

c. $64^{\frac{1}{6}} = \sqrt[6]{64} = 2$

d. $(-32)^{\frac{1}{4}}$ is undefined in $\mathbb{R}$

e. $(-32)^{\frac{1}{5}} = \sqrt[5]{-32} = -2$

f. $(32)^{-\frac{1}{5}} = \frac{1}{32^{\frac{1}{5}}} = \frac{1}{\sqrt[5]{32}} = \frac{1}{2}$

Now that we have defined $a^{\frac{1}{n}}$, we can use the properties of exponents to define $a^{\frac{m}{n}}$ for any rational number $\frac{m}{n}$:

$$\boxed{a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = (a^{\frac{1}{n}})^m \text{ or } a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m}.$$

EXAMPLE 6 Evaluate each expression.

a. $27^{\frac{2}{3}}$ b. $32^{\frac{3}{5}}$ c. $64^{-\frac{2}{3}}$ d. $25^{\frac{3}{2}}$

Solution a. $27^{\frac{2}{3}}=(27^{\frac{1}{3}})^2=3^2=9$

b. $32^{\frac{3}{5}}=(32^{\frac{1}{5}})^3=2^3=8$

c. $64^{-\frac{2}{3}}=\dfrac{1}{64^{\frac{2}{3}}}=\dfrac{1}{(64^{\frac{1}{3}})^2}=\dfrac{1}{4^2}=\dfrac{1}{16}$

d. $25^{\frac{3}{2}}=(25^{\frac{1}{2}})^3=5^3=125$

$3^3=27 \Leftrightarrow 27^{\frac{1}{3}}=3$

$2^5=32 \Leftrightarrow 32^{\frac{1}{5}}=2$

$4^3=64 \Leftrightarrow 64^{\frac{1}{3}}=4$

$5^2=25 \Leftrightarrow 25^{\frac{1}{2}}=5$

EXAMPLE 7 Evaluate the expressions.

a. $4^{-\frac{1}{2}}-8^{-\frac{2}{3}}+16^{-\frac{3}{4}}$ b. $(0.0016)^{\frac{1}{4}}+(0.125)^{\frac{1}{3}}$

Solution a. $4^{-\frac{1}{2}}-8^{-\frac{2}{3}}+16^{-\frac{3}{4}}=\dfrac{1}{4^{\frac{1}{2}}}-\dfrac{1}{8^{\frac{2}{3}}}+\dfrac{1}{16^{\frac{3}{4}}}=\dfrac{1}{2}-\dfrac{1}{(8^{\frac{1}{3}})^2}+\dfrac{1}{(16^{\frac{1}{4}})^3}=\dfrac{1}{2}-\dfrac{1}{2^2}+\dfrac{1}{2^3}=\dfrac{4-2+1}{8}=\dfrac{3}{8}$

b. $(0.0016)^{\frac{1}{4}}+(0.125)^{\frac{1}{3}}=\left(\dfrac{16}{10000}\right)^{\frac{1}{4}}+\left(\dfrac{125}{1000}\right)^{\frac{1}{3}}=\dfrac{16^{\frac{1}{4}}}{10000^{\frac{1}{4}}}+\dfrac{125^{\frac{1}{3}}}{1000^{\frac{1}{3}}}=\dfrac{2}{10}+\dfrac{5}{10}=\dfrac{7}{10}$

$2^3=8 \Leftrightarrow 8^{\frac{1}{3}}=2$

$2^4=16 \Leftrightarrow 16^{\frac{1}{4}}=2$

$5^3=125 \Leftrightarrow 125^{\frac{1}{3}}=5$

EXAMPLE 8 Write each expression as a quotient with positive integer exponents, given that all variables represent positive real numbers.

a. $(x^{-4}\cdot y^{12})^{\frac{3}{4}}$ b. $(x^{-\frac{4}{3}}\cdot y^{-\frac{1}{4}})(x^{\frac{1}{3}}\cdot y^{\frac{5}{4}})$

Solution a. $(x^{-4}\cdot y^{12})^{\frac{3}{4}}=(\dfrac{y^{12}}{x^4})^{\frac{3}{4}}=\dfrac{(y^{12})^{\frac{3}{4}}}{(x^4)^{\frac{3}{4}}}=\dfrac{y^{12\cdot\frac{3}{4}}}{x^{4\cdot\frac{3}{4}}}=\dfrac{y^9}{x^3}$

b. $(x^{-\frac{4}{3}}\cdot y^{-\frac{1}{4}})(x^{\frac{1}{3}}\cdot y^{\frac{5}{4}})=x^{-\frac{4}{3}}\cdot x^{\frac{1}{3}}\cdot y^{-\frac{1}{4}}\cdot y^{\frac{5}{4}}=x^{-\frac{4}{3}+\frac{1}{3}}\cdot y^{-\frac{1}{4}+\frac{5}{4}}=x^{-1}\cdot y=\dfrac{y}{x}$

EXAMPLE 9 Factorize each expression using the given common factor, given that x represents a positive real number.

a. $3x^{\frac{5}{4}}-x^{\frac{1}{4}};\ x^{\frac{1}{4}}$ b. $9x^{\frac{3}{5}}+21x^{-\frac{1}{5}};\ 3x^{-\frac{2}{5}}$

Solution a. We need $x^{\frac{1}{4}}$ as a factor in each term of $3x^{\frac{5}{4}} - x^{\frac{1}{4}}$. Look at the exponents. Since $\frac{5}{4} = 1 + \frac{1}{4}$, we can write

$$3x^{1+\frac{1}{4}} - x^{\frac{1}{4}} = 3x \cdot x^{\frac{1}{4}} - x^{\frac{1}{4}} = x^{\frac{1}{4}} \cdot (3x - 1).$$

b. $$9x^{\frac{3}{5}} + 21x^{-\frac{1}{5}} = \left[3 \cdot 3x^{1-\frac{2}{5}}\right] + \left[3 \cdot 7x^{\frac{1}{5}-\frac{2}{5}}\right] = \left[3x \cdot 3x^{-\frac{2}{5}}\right] + \left[7x^{\frac{1}{5}} \cdot 3x^{-\frac{2}{5}}\right] = 3x^{-\frac{2}{5}} \cdot (3x + 7x^{\frac{1}{5}})$$

Check Yourself 3

1. Evaluate the expressions. a. $-25^{\frac{1}{2}}$ b. $100^{\frac{3}{2}}$
2. Write each expression as a quotient with positive integer exponents, given that all variables represent positive real numbers.

   a. $(x^{-4} \cdot y^{8})^{-\frac{3}{4}}$ b. $(-27a^{-6} \cdot b^{12})^{-\frac{1}{3}}$
3. Factorize each expression using the given common factor, given that x represents a positive real number.

   a. $x^{\frac{4}{3}} + 3x^{\frac{1}{3}};\ x^{\frac{1}{3}}$ b. $35x^{-\frac{5}{4}} - 14x^{\frac{3}{4}};\ 7x^{-\frac{1}{4}}$

Answers

1. a. -5 b. 1000 2. a. $\dfrac{x^3}{y^6}$ b. $-\dfrac{a^2}{3b^4}$ 3. a. $x^{\frac{1}{3}}(x+3)$ b. $7x^{-\frac{1}{4}}(\dfrac{5}{x} - 2x)$

D. REAL EXPONENTS

So far we have studied expressions with rational exponents. What about expressions with irrational exponents, such as $3^{\sqrt{2}}$ or 2^{π}? Many scientific calculations include expressions such as these. We will not study irrational exponents in detail in this module. Instead, we will simply say that the laws we have seen for rational exponents also hold for irrational exponents.

If a is a positive number and x is an irrational number, we can evaluate a^x by taking successive rational approximations of x. For example, to evaluate $3^{\sqrt{2}}$ we round approximations of $\sqrt{2} = 1.41421356...$ up and down and then raise 3 to these rational powers, as shown in the box.

$$\begin{gathered} 3^{1} < 3^{\sqrt{2}} < 3^{2} \\ 3^{1.4} < 3^{\sqrt{2}} < 3^{1.5} \\ 3^{1.41} < 3^{\sqrt{2}} < 3^{1.42} \\ 3^{1.414} < 3^{\sqrt{2}} < 3^{1.415} \\ 3^{1.4142} < 3^{\sqrt{2}} < 3^{1.4143} \\ \vdots \end{gathered}$$

As the process continues, the left and right sides of the inequality expressed as rational powers of 3 will have more and more identical decimal digits. These are the digits in the decimal approximation to $\sqrt{2}$. At the fourth step, we can use $3^{\sqrt{2}} \approx 3^{1.414}$.

Using a scientific calculator, we find $3^{1.414} \approx 4.727695$.

The actual value of $3^{\sqrt{2}}$ is 4.728804... .

We will not go into any more detail here, but it is easy to see that a^x has a very definite real value for $a > 0$ and for any real (i.e. rational or irrational) value of x.

EXAMPLE 10 Evaluate the expressions for $x \in \mathbb{R}$ given that $3^{-x} + 3^{x} = 5$.

a. $9^{x} + 9^{-x}$ b. $(\frac{1}{3})^{x} + (\frac{1}{3})^{-x}$

Solution a. We begin with $3^{-x} + 3^{x} = 5$ and try to obtain the expression $9^{x} + 9^{-x}$. Taking the square of each side of the equation gives us

$(a + b)^2 = a^2 + b^2 + 2ab$
$(a^m)^n = (a^n)^m$

$$(3^{-x} + 3^{x})^2 = 5^2 \Leftrightarrow (3^{-x})^2 + (3^{x})^2 + (2 \cdot 3^{x} \cdot 3^{-x}) = 25 \Leftrightarrow (3^2)^{-x} + (3^2)^{x} + (2 \cdot 3^{x+(-x)}) = 25$$

$$\Leftrightarrow 9^{-x} + 9^{x} + (2 \cdot \underbrace{3^0}_{1}) = 25 \Leftrightarrow 9^{x} + 9^{-x} = 23.$$

b. Using $\frac{1}{3} = 3^{-1}$, we get $(\frac{1}{3})^{x} + (\frac{1}{3})^{-x} = (3^{-1})^{x} + (3^{-1})^{-x} = 3^{-x} + 3^{x} = 5.$

EXERCISES 1.1

A. Integer Exponents

1. Evaluate the expressions.

a. 9^{-1} b. 3^{-2}

c. $(\frac{1}{2})^{-1}$ d. $(\frac{3}{2})^{-2}$

e. $8 \cdot 2^{-3}$ f. $3 \cdot 4^{0}$

g. $9 \cdot 2^{-2}$ h. $(9 \cdot 3)^{-2}$

i. $2^{-1} + 3^{-1}$ j. $(3^{-2} + 2^{-2})^{-1}$

k. $2^{-1} + 4^{-1} + 8^{-1}$ l. $\dfrac{-3^4 \cdot (-3)^5}{3^{-5} \cdot (-3)^{-5}}$

✪m. $\dfrac{9^{94} - 3^{94}}{3^{141} + 3^{94}} + 1$

2. Simplify each expression, given that all variables are positive. Leave each answer as a single expression with positive integer exponents.

a. $a^3 \cdot a^5$ b. $(a^{-2})^4$ c. $(x^{-2})^{-4}$

d. $((x^{-2})^{-1})^{-3}$ e. $(2p^{-3})^{-1}$ f. $(-4x^{-3})^{-2}$

g. $((3x)^{-1})^{-2}$ h. $3(x^2y)^2$ i. $\dfrac{x^5}{x^3}$

j. $\dfrac{x^2}{x^5}$ k. $\dfrac{x^8}{x^{-3}}$ l. $\dfrac{x^{-3} \cdot x^{-2}}{x^5}$

B. Roots and Radical Expressions

3. Evaluate each radical expression.

a. $\sqrt[12]{27} \cdot \sqrt[4]{27}$ b. $\sqrt[4]{81}$ c. $\sqrt[3]{\sqrt{64}}$

4. Simplify each expression, given that all variables represent positive numbers.

a. $\sqrt[4]{81x^{12} \cdot y^{-8}}$ b. $\sqrt[3]{x} \cdot \sqrt[4]{x^3}$ c. $\sqrt[4]{9x^6}$

d. $\sqrt[6]{\dfrac{1}{8x^{24}}}$ ✪e. $\dfrac{x\sqrt{y} + y\sqrt{x}}{\sqrt{x} + \sqrt{y}}$ f. $\sqrt[3]{-0.125x^3y^{-15}}$

C. Rational Exponents

5. Evaluate the expressions.

a. $9^{\frac{1}{2}}$ b. $9^{-\frac{1}{2}}$ c. $4^{\frac{3}{2}}$

d. $16^{-\frac{3}{4}}$ e. $-25^{\frac{1}{2}}$ f. $(-9)^{-\frac{1}{2}}$

g. $32^{\frac{1}{5}}$ h. $8^{-\frac{1}{3}}$ i. $(-8)^{\frac{1}{3}}$

j. $(\frac{4}{25})^{-\frac{1}{2}}$ k. $(-\frac{27}{8})^{\frac{1}{3}}$ l. $(36^{-1})^{\frac{1}{2}}$

m. $((\frac{1}{4})^{-3})^{\frac{1}{12}}$ n. $64^{\frac{2}{5}} \cdot 2^{-\frac{2}{5}}$ o. $3^{\frac{1}{2}} \cdot 27^{\frac{1}{2}}$

6. Simplify each expression, given that all variables represent positive real numbers.

a. $(243a^5)^{-\frac{1}{5}}$ b. $(8m^{-6})^{\frac{1}{3}}$ c. $(a^4 \cdot a^{-8})^{-\frac{1}{2}}$

d. $(9n^{-4})^{\frac{1}{2}}$ e. $2x^{\frac{4}{3}} \cdot 4x^{-\frac{1}{3}}$ f. $3x^{\frac{1}{2}} \cdot 4x^{-\frac{3}{2}}$

D. Real Exponents

7. Evaluate each expression.

a. $3^{\sqrt{2}} \cdot 3^{-\sqrt{2}}$ b. $\dfrac{5^{\sqrt{3}}}{5^{-\sqrt{3}}}$ c. $(4^{\sqrt{3}})^{-\sqrt{3}}$

d. $\left[(\frac{1}{2})^{\sqrt{2}}\right]^{\frac{1}{\sqrt{2}}}$ e. $\dfrac{2^{\sqrt{12}}}{2^{\sqrt{48}}}$ f. $(6^{-\frac{2}{\pi}})^{\pi}$

8. Simplify the expressions.

✪a. $\dfrac{a^{x+3} + a^{x+1} + a^{x-1}}{a^{x-5} + a^{x-3} + a^{x-1}}$

✪b. $\dfrac{m}{(\frac{n}{m})^{n} + 1} + \dfrac{m}{(\frac{n}{m})^{-n} + 1}$

9. ✪ Evaluate the expressions for $x \in \mathbb{R}$, given that $2^x + 2^{-x} = 3$.

a. $4^x + 4^{-x}$ b. $8^x + 8^{-x}$

The Number e

The history of mathematics is marked by the discovery of special numbers such as counting numbers, zero, negative numbers, and imaginary numbers. One of the most famous numbers of modern times is called e, or the *Euler number*.

John Napier, the inventor of logarithms, implied that this number existed in his work at the beginning of the seventeenth century. However, the Swiss mathematician Euler studied it more formally in the 1720s, and in 1727 he gave it the name e that we use today.

e is a real number constant that appears in different areas of mathematics. It is important in statistics for calculating the change in size of a population or quantity over time. It helps us to solve problems in probability and counting, and appears in the study of the distribution of prime numbers. We also use it to solve problems with logarithmic or exponential functions in calculus.

So how can we calculate this important number e?

One way is to define e as the number that the expression $\left(1+\frac{1}{n}\right)^n$ approaches as n tends to infinity.

n	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10000	2.71815
100000	2.71827
1000000	2.71828
1000000000	2.718281828

The table below shows the value of this expression for different values of n. You can see from the table that as n gets larger and larger, the expression $\left(1+\frac{1}{n}\right)^n$ gets closer and closer to 2.71828..., which is the value of e. In fact, its value is approximately 2.718 281 828 459 045 235 360 287 471.

Alternatively, we can calculate the value of e using the following infinite sum: $e=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots$.

Recently, mathematicians have discovered even more efficient ways of calculating e.

e is an irrational number, so its decimal expansion never terminates or completely repeats. Therefore, no matter how many digits in the expansion of e you know, the only way to find the next digit is to compute e using a particular method with more accuracy.

e is also a transcendental number, which means that e is not the root of any polynomial with rational number coefficients.

Finally, e is also the base of special types of logarithm known as *natural logarithms*. You will study natural logarithms later in this module.

2 EXPONENTIAL FUNCTIONS

Imagine that a virus is spreading through the population of your city. The virus begins in one person, and spreads to all the people that the person meets. In turn, these people infect all the people that they meet, and so on. Traveling like this, the virus can spread very fast.

Mathematicians use special functions called exponential functions to model and study situations such as this.

In this section we will look at simple exponential functions which we call basic exponential functions. In the last section of this chapter, we will look at the more general form of an exponential function.

A. BASIC CONCEPT

Definition

basic exponential function

A function of the form $f(x) = a^x$ for a constant $a > 0$, $a \neq 1$ is called a basic exponential function with base a.

In this section, the term 'exponential function' means a basic exponential function. $g(x) = 3^x$, $h(x) = 10^x$ and $j(x) = 4^{2x} = (4^2)^x = 16^x$ are examples of basic exponential functions. $m(x) = (-2)^x$ and $n(x) = x^5$ are not basic exponential functions (can you see why?).

EXAMPLE 11 Determine which functions are exponential.

a. $p(x) = 2^{2x}$ b. $r(x) = 3^{-x}$ c. $s(x) = 5^{\frac{x}{4}}$ d. $t(x) = x^3$ e. $u(x) = (-3)^x$

Solution We can use the laws of exponents to identify the functions which satisfy the definition.

a. $p(x) = 2^{2x} = (2^2)^x = 4^x$. Since the base (4) is positive and different from 1, $p(x)$ is an exponential function.

b. $r(x) = 3^{-x} = (3^{-1})^x = (\frac{1}{3})^x$. Since $\frac{1}{3} > 0$ and $\frac{1}{3} \neq 1$, $r(x)$ is also an exponential function.

c. $s(x) = 5^{\frac{x}{4}} = (5^{\frac{1}{4}})^x = (\sqrt[4]{5})^x$. Therefore, $s(x)$ is an exponential function with base $\sqrt[4]{5}$.

d. Since x appears in the base and not in the exponent, $t(x)$ is not an exponential function.

e. Since $u(x)$ has a negative base, u is not an exponential function.

B. GRAPHS OF EXPONENTIAL FUNCTIONS

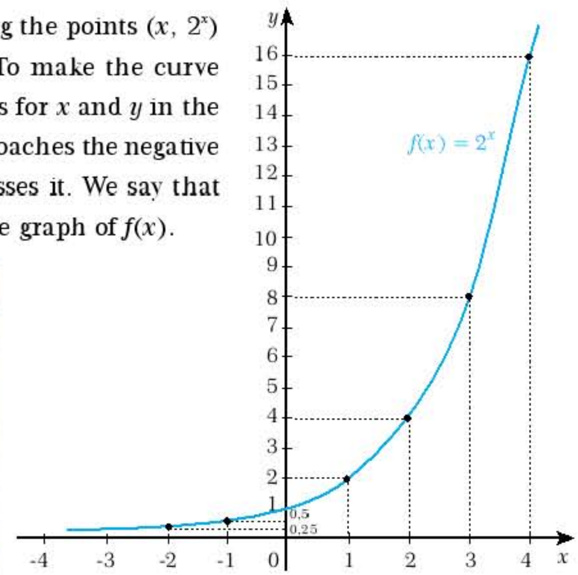
Let us look at the graph of a basic exponential function for the two possible cases $a > 1$ and $a \in (0, 1)$.

1. Graph of $f(x) = a^x$ for $a > 1$

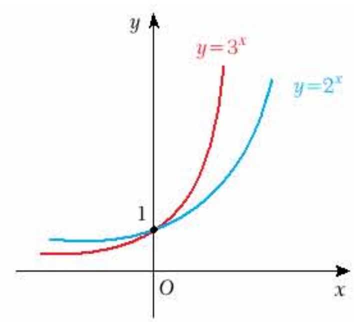
Consider the function $f(x) = 2^x$. The following table shows the corresponding values of x and $f(x)$.

x	$-\infty$	-4	-3	-2	-1	0	1	2	3	4	$+\infty$
$f(x)=2^x$		$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	

We can sketch the graph of $f(x)$ by plotting the points $(x, 2^x)$ and joining them with a smooth curve. To make the curve easier to see, we have used different scales for x and y in the figure opposite. Notice that the curve approaches the negative x-axis but never actually touches it or crosses it. We say that the x-axis is a horizontal asymptote for the graph of $f(x)$.



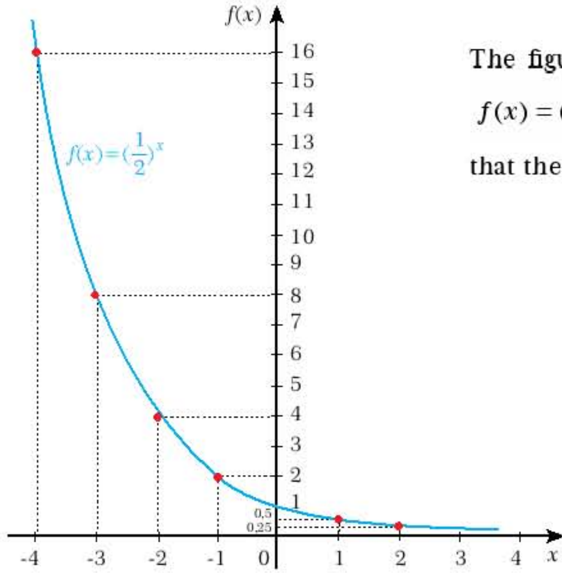
The graph of any exponential function with a base $a > 1$ is a curve similar to the one shown above. The figure opposite shows the graphs $y = 2^x$ and $y = 3^x$. We can see that as the the base a increases, the graph $y = a^x$ becomes steeper on the right and approaches the asymptote faster on the left.



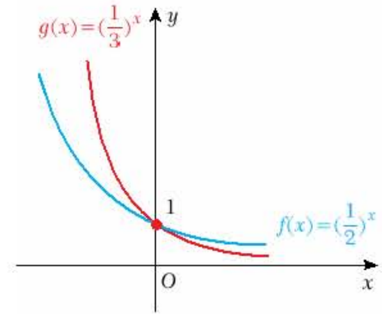
2. Graph of $f(x) = a^x$ for $a \in (0, 1)$

Consider the function $f(x) = (\frac{1}{2})^x$. The table shows a sample of values.

x	$-\infty$	-4	-3	-2	-1	0	1	2	3	4	5	$+\infty$
$f(x)=(\frac{1}{2})^x$		16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	



The figure on the left shows a sketch of the graph $f(x) = (\frac{1}{2})^x$ for the values in the table. We can see that the x-axis is a horizontal asymptote of the curve.



If we compare the graphs of $f(x) = (\frac{1}{2})^x$ and $g(x) = (\frac{1}{3})^x$ shown on the right, we can conclude that as the base $a \in (0, 1)$ decreases, the graph $y = a^x$ becomes steeper on the left and approaches the asymptote faster on the right.

Notice that the graph $y = 2^x$ is a reflection of the graph $y = (\frac{1}{2})^x$ in the y-axis.

C. PROPERTIES OF EXPONENTIAL FUNCTIONS

Any exponential function of the form $f(x) = a^x$ has the following properties:

1. The domain of the function is the infinite interval $(-\infty, +\infty)$.

   In other words, x can take any value in $\mathbb{R}$.

2. The range of the function is the set of positive real numbers.

   In other words, the function $f(x) = a^x$ is positive for all values of the argument x, and its graph lies entirely above the x-axis.

   Combining properties 1 and 2 gives us f: $(-\infty, +\infty) \to (0, \infty)$, i.e. f: $\mathbb{R} \to \mathbb{R}^+$.

3. Exponential functions are monotone.

   a. If $a > 1$ then $f(x) = a^x$ is strictly increasing because the curve of its graph always moves upward as x increases. In other words,

   if $x_1 < x_2$ then $a^{x_1} < a^{x_2}$, and if $a^{x_1} < a^{x_2}$ then $x_1 < x_2$, where $a > 1$.

   For example, since $3 < 4$ we can write $5^3 < 5^4$.

The **domain** of a function $f(x)$ is the set of values of x for which $f(x)$ is defined.
The **range** of a function $f(x)$ is the set of possible values which $f(x)$ can take.
A function is called **monotone** if it is either increasing or decreasing.

b. If $0 < a < 1$ then $f(x) = a^x$ is strictly decreasing because the curve of its graph always moves downward as x increases. In other words,

$$\boxed{\text{if } x_1 < x_2 \text{ then } a^{x_1} > a^{x_2}, \text{ and if } a^{x_1} < a^{x_2} \text{ then } x_1 > x_2, \text{ where } 0 < a < 1.}$$

We can remember relations a and b by using a simple rule:

Consider the direction of the inequality between the exponents. If the base is greater than 1, the direction of the inequality between the expressions is the same as this direction. If the base is between zero and 1, the direction of the inequality is opposite to this direction.

For example, to compare $2^{1/2}$ and $2^{3/4}$, we first establish the inequality between their exponents. Since their common base is greater than 1, we keep the direction of the inequality the same for the two numbers:

$\underbrace{\frac{1}{2} < \frac{3}{4} \Leftrightarrow 2^{\frac{1}{2}} < 2^{\frac{3}{4}}}_{\text{the same}}$. Alternatively, by the same rule we get $\underbrace{5 < 7 \Leftrightarrow (\frac{2}{3})^5 > (\frac{2}{3})^7}_{\text{opposite}}$, since $0 < \frac{2}{3} < 1$.

4. The graph of the function of the form $f(x) = a^x$ always cuts the y-axis at $y = 1$, since at $x = 0$, $f(x) = a^x = 1$.

For other values of x we can write the following:

- For $x > 0$, $a^x > 1$ if $a > 1$ and $a^x \in (0, 1)$ if $a \in (0, 1)$.
- For $x < 0$, $a^x \in (0, 1)$ if $a > 1$ and $a^x > 1$ if $a \in (0, 1)$.

EXAMPLE 12 Identify which exponential numbers are greater than 1.

a. $(\frac{1}{2})^{0.4}$ b. $(1.5)^{1.4}$ c. $(\sqrt{3})^{-\sqrt{2}}$ d. $(\frac{1}{\sqrt{5}})^{\frac{1}{2}}$ e. $(\frac{2}{3})^{1.5}$ f. $(0.1\overline{2})^{-0.5}$

Solution Combining properties 3 and 4 above gives us the following:

- For $0 < a < 1$, if $x > 0$ then $0 < a^x < 1$, and if $x < 0$ then $a^x > 1$.
- For $a > 1$, if $x > 0$ then $a^x > 1$, and if $x < 0$ then $0 < a^x < 1$.

By considering these statements, we can see that the numbers in b and f are greater than 1.

EXAMPLE 13 Find the biggest number in each pair.

a. $5^{\frac{3}{4}}, 5^{\frac{4}{5}}$ b. $(\frac{2}{3})^{\frac{5}{2}}, (\frac{2}{3})^3$ c. $(\sqrt{2})^{-2}, (\frac{1}{\sqrt{2}})^3$

Solution a. Since $\frac{3}{4} < \frac{4}{5}$ and the common base is greater than 1, the direction of the inequality will be the same for the exponential expressions, i.e. $5^{\frac{3}{4}} < 5^{\frac{4}{5}}$. So $5^{\frac{4}{5}}$ is the biggest.

b. The common base $(\frac{2}{3})$ is between zero and 1. So the direction of the inequality for the exponential expressions will be the opposite of the inequality for the exponents. Since $\frac{5}{2} < 3$, we can conclude $(\frac{2}{3})^{\frac{5}{2}} > (\frac{2}{3})^3$.

c. $(\sqrt{2})^{-2} = (\frac{1}{\sqrt{2}})^2$ and $0 < \frac{1}{\sqrt{2}} < 1$. So $(\frac{1}{\sqrt{2}})^2 > (\frac{1}{\sqrt{2}})^3$ because $2 < 3$.

EXAMPLE 14 State the monotony (strictly increasing or strictly decreasing) of each function $f : \mathbb{R} \to \mathbb{R}$.

a. $f(x) = 2^x + 3^x$ b. $f(x) = (\frac{1}{2})^x + (\frac{1}{3})^x$ c. $f(x) = 1 + \frac{1}{2^x}$

d. $f(x) = 2^{2+x}$ e. $f(x) = -3 \cdot 2^{-x}$

Solution Recall the properties for the monotony of the sum of two functions:

1. The sum of two strictly increasing functions is also a strictly increasing function.
2. The sum of two strictly decreasing functions is also a strictly decreasing function.

a. Since $f_1(x) = 2^x$ and $f_2(x) = 3^x$ are both strictly increasing, $f(x) = f_1(x) + f_2(x) = 2^x + 3^x$ is also strictly increasing.

b. Since both $(\frac{1}{2})^x$ and $(\frac{1}{3})^x$ are strictly decreasing, $f(x) = (\frac{1}{2})^x + (\frac{1}{3})^x$ is also strictly decreasing.

c. Adding a constant to a monotone function does not affect the monotony of the resulting function. We can conclude that since $(\frac{1}{2})^x$ is strictly decreasing, $f(x) = 1 + \frac{1}{2^x} = 1 + (\frac{1}{2})^x$ is strictly decreasing.

d. We can write $f(x) = 2^{2+x} = 2^2 \cdot 2^x = 4 \cdot 2^x$ and notice that 2^x is a strictly increasing function. Since a positive constant multiple of a function has the same monotony as this function, $f(x) = 2^{2+x} = 4 \cdot 2^x$ is a strictly increasing function.

e. $f(x) = -3 \cdot 2^{-x} = -3 \cdot (\frac{1}{2})^x$. Since a negative constant multiple of a function has the opposite monotony of this function and $(\frac{1}{2})^x$ is strictly decreasing, $f(x) = -3 \cdot (\frac{1}{2})^x$ is strictly increasing.

A bijective function is a function which is both one-to-one and onto.

A fifth and final property of exponential functions is that they are bijective. In particular, since one-to-one functions satisfy the property $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$, we can draw the following conclusion about exponential numbers:

If $a^x = a^y$ then $x = y$, and if $x = y$ then $a^x = a^y$.

For example, $x = 5$ is the solution to the equation $2^x = 2^5$.

EXAMPLE 15 In each case, find a function of the form $f(x) = m \cdot a^x$ which satisfies the conditions.

a. $f(0) = 2$ and $f(3) = 250$ b. $f^{-1}(20) = 2$ and $f^{-1}(40) = 3$

Solution a. Substituting the two function values in the required form of $f(x)$ gives us

$$f(0) = 2 \Leftrightarrow m \cdot \underbrace{a^0}_{1} = 2 \Leftrightarrow m = 2, \text{ and } f(3) = 250 \Leftrightarrow \underbrace{m}_{2} \cdot a^3 = 250 \Leftrightarrow a = 5.$$

So the required function is $f(x) = 2 \cdot 5^x$.

b. Considering the relation between a function and its inverse, we have

$$f^{-1}(20) = 2 \Leftrightarrow f(2) = 20, \text{ and } f^{-1}(40) = 3 \Leftrightarrow f(3) = 40.$$

$f(a) = b \Leftrightarrow f^{-1}(b) = a$

Substituting these values in the required form for $f(x)$ gives us

$$f(2) = 20 \Leftrightarrow m \cdot a^2 = 20, \text{ and } f(3) = 40 \Leftrightarrow m \cdot a^3 = 40 \Leftrightarrow m \cdot a^2 \cdot a = 40.$$

Now let us replace $m \cdot a^2 = 20$ in the second equation:

$$20 \cdot a = 40 \Leftrightarrow a = 2.$$

Substituting this in the second equation again gives us

$$m \cdot a^3 = 40 \Leftrightarrow m \cdot 2^3 = 40, \text{ i.e. } m = 5.$$

So the function is $f(x) = 5 \cdot 2^x$.

Check Yourself 4

1. Find the biggest number in each pair.

   a. $3^{\frac{2}{5}}, 3^{\frac{1}{3}}$ b. $(\frac{1}{4})^2, (0.25)^3$ c. $(\sqrt[3]{25})^4, (5)^{\sqrt{3}+1}$

2. Determine whether each number is greater than 1.

   a. $(\frac{1}{3})^{\frac{1}{2}}$ b. $2^{\frac{3}{2}}$ c. $(\frac{1}{2})^{3-\sqrt{3}}$ d. $(\frac{\pi}{3})^{1-\sqrt{2}}$

3. Determine whether each function is increasing (↗) or decreasing (↘).

   a. $f(x) = 2^{1-x}$ b. $f(x) = 3^x + 5^x$ c. $f(x) = \frac{2^x + 3^x}{5^x}$ d. $f(x) = 1 - 2^x$

4. In each case, find a function of the form $f(x) = m \cdot a^x$ which satisfies the conditions.

   a. $f(1) = -\frac{2}{9}$ and $f(2) = -\frac{2}{27}$ b. $f^{-1}(-\frac{27}{4}) = 2$ and $f^{-1}(-\frac{81}{8}) = 3$

Answers

1. a. $3^{\frac{2}{5}}$ b. $(\frac{1}{4})^2$ c. $5^{\sqrt{3}+1}$
2. a. no b. yes c. no d. no
3. a. ↘ b. ↗ c. ↘ d. ↗
4. a. $f(x) = -\frac{2}{3} \cdot (\frac{1}{3})^x$ b. $f(x) = -3 \cdot (\frac{3}{2})^x$

EXERCISES 1.2

A. Basic Concept

1. Determine which functions are exponential functions.

a. $f(x) = 5^{3x}$ b. $f(x) = 4^{-2x}$

c. $f(x) = 3^{x-2}$ d. $f(x) = -2^{x}$

e. $f(x) = x^{x}$ f. $f(x) = x^{2}$

g. $f(x) = (-2)^{x}$

B. Graphs of Exponential Functions

2. Sketch the graph of each function.

a. $f(x) = 3^{x}$ b. $f(x) = (\frac{1}{3})^{x}$

c. $f(x) = 5^{x}$ d. $f(x) = (\frac{1}{5})^{x}$

e. $f(x) = 2^{2x}$ f. $f(x) = 3^{-x}$

C. Properties of Exponential Functions

3. In each case, find a function of the form $f(x) = m \cdot a^{x}$ which satisfies the conditions.

a. $f(0) = 3$ and $f(2) = 12$

b. $f(1) = \frac{3}{2}$ and $f(4) = \frac{3}{16}$

c. $f^{-1}(27) = 3$ and $f^{-1}(81) = 4$

4. Which number in each pair is greater?

a. $4^{\frac{3}{4}}, 4^{\frac{7}{8}}$ b. $(\frac{4}{5})^{10}, (0.8)^{\frac{21}{2}}$

c. $(0.7)^{\frac{2}{3}}, (\frac{7}{10})^{\frac{25}{36}}$ d. $(\sqrt{6})^{-6}, (\frac{1}{\sqrt{6}})^{0.6}$

e. $(\sqrt{2}-1)^{\sqrt{2}+1}, \sqrt[5]{(\sqrt{2}-1)^{12}}$

✪f. $x = (\sqrt{5})^{\frac{2}{\sqrt{3}-1}}, y = \sqrt{5} \cdot (\frac{25}{5^{\frac{7}{2}}})^{-1}$

5. Determine whether each number is greater than 1.

a. $(\frac{\pi}{3})^{\sqrt{5}-3}$ b. $(\frac{\pi-1}{2})^{\sqrt{2}}$ c. $\frac{5^{\sqrt{27}}}{5^{\sqrt{3}}} \cdot 5^{-\sqrt{12}}$

d. $(\sqrt{5}-2)^{\sqrt{2}-2}$ e. $(\sqrt{2}-1)^{0.1}$ f. $(\sqrt{6}-2)^{-2}$

6. ✪ Determine whether each function is increasing (↗) or decreasing (↘).

a. $f(x) = 3^{x+1}$ b. $f(x) = 2^{1-2x}$

c. $f(x) = 5 \cdot 3^{x-3}$ d. $f(x) = (\frac{2}{3})^{x} + (\frac{3}{5})^{x}$

e. $f(x) = \frac{3^{x}+5^{x}}{2^{x}}$ f. $f(x) = 1 - \frac{3}{4^{x}}$

7. ✪✪ For $a > 1$ and $x, y, z \in \mathbb{R}$, show that if $x + y > 2z$ then $a^{x} + a^{y} > 2a^{z}$.

3 SIMPLE VARIATIONS OF EXPONENTIAL FUNCTIONS (OPTIONAL)

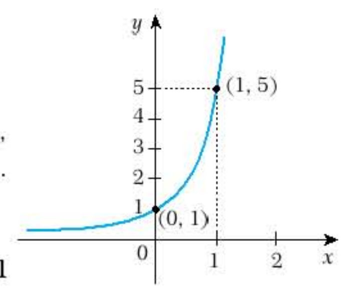
Earlier in this chapter we looked at the graphs of basic exponential functions of the form $f(x) = a^x$ and their properties. We sketched the graphs of some functions by plotting a selection of points, and we drew the following conclusions about exponential functions and their graphs:

1. The domain is $\mathbb{R}$, i.e. the graph is defined over the x-axis.
2. The range is $\mathbb{R}^+$, i.e. $f(x)$ always has positive values.
3. The graph does not cross the x-axis, and has a horizontal asymptote at $y = 0$ (the x-axis).
4. As a result, there are no x-intercepts.
5. The graph crosses the y-axis at (0, 1), i.e. $y = 1$ is the y-intercept.
6. $(1, a)$ is also a point on the graph because $f(1) = a^1 = a$.
7. $f(x)$ is a monotone function which is strictly increasing for $a > 1$ and strictly decreasing for $a \in (0, 1)$.

Considering all these common properties, we can roughly sketch the graph of any exponential function of the form $f(x) = a^x$.

EXAMPLE 16 Sketch the graph of $f(x) = 5^x$.

Solution We know that the points (0, 1) and (1, 5) lie on the graph. Also, $f(x) = 5^x$ is strictly increasing because its base, 5, is greater than 1. We can therefore sketch the graph as shown opposite.



Now we are ready to consider the graphs of more general exponential functions.

Definition **exponential function (general form)**

A function of the form $f(x) = c \cdot a^{d(x+p)} + k$ where a, c, d, p and k are real numbers with $a > 0$ and $a \neq 1$ is called an exponential function with base a.

For example, $g(x) = 3^{-x}$, $h(x) = 10^{x+1}$ and $j(x) = 5^{-3x} + 2$ are all exponential functions.

Before we try to sketch the graph of an exponential function, it is useful to remember how certain variations of any function $f(x)$ affect its graph. Consider the function $f(x)$ and the positive contants c, d, k and p. The possible transformations of the graph of $f(x)$ are as follows:

Vertical shift: We obtain the graph $y = f(x) + k$ by shifting $f(x)$ k units upward, and the graph of $y = f(x) - k$ by shifting $f(x)$ k units downward.

Horizontal shift: We obtain the graph $y = f(x + p)$ by shifting $f(x)$ p units to the left, and the graph $y = f(x - p)$ by shifting $f(x)$ p units to the right.

Reflection: The graph of $y = f(-x)$ is obtained by reflecting $f(x)$ in the y-axis, and the graph of $y = -f(x)$ is obtained by reflecting $f(x)$ in the x-axis.

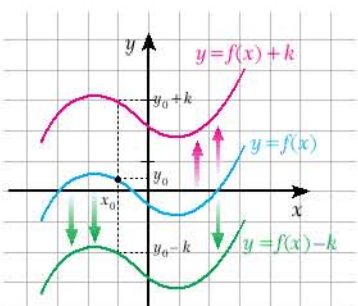
Vertical shrink and stretch: We obtain the graph of $y = c \cdot f(x)$ by shrinking $f(x)$ vertically by a factor of c if $0 < c < 1$, and by stretching $f(x)$ vertically by a factor of c if $c > 1$.

Horizontal shrink and stretch: We obtain the graph $y = f(d \cdot x)$ by shrinking $f(x)$ horizontally by a factor of $\dfrac{1}{d}$ if $d > 1$, and by stretching it horizontally by the same factor if $0 < d < 1$.

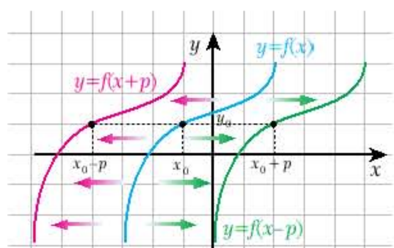
In summary, each point (x, y) on the graph of $f(x)$ moves to

- $(x, y + k)$ for $y = f(x) + k$ (shift k units up)
- $(x, y - k)$ for $y = f(x) - k$ (shift k units down)
- $(x - p, y)$ for $y = f(x + p)$ (shift p units to the left)
- $(x + p, y)$ for $y = f(x - p)$ (shift p units to the right)
- $(-x, y)$ for $y = f(-x)$ (reflection in the y-axis)
- $(x, -y)$ for $y = -f(x)$ (reflection in the x-axis)
- $(x, c \cdot y)$ for $y = c \cdot f(x)$ (vertical shrink or stretch)
- $(\frac{1}{d} \cdot x, y)$ for $y = f(x \cdot d)$ (horizontal shrink or stretch).

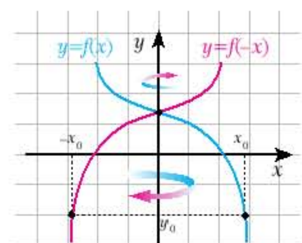
The following figures show the effects of each variation on the graph $f(x)$.



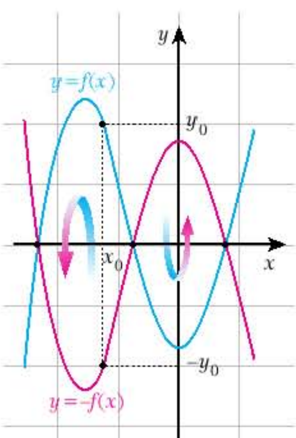
vertical shift



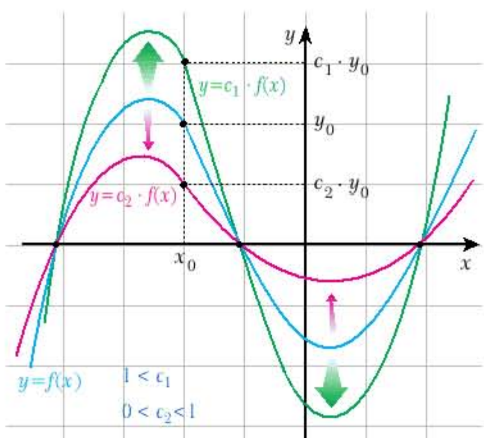
horizontal shift



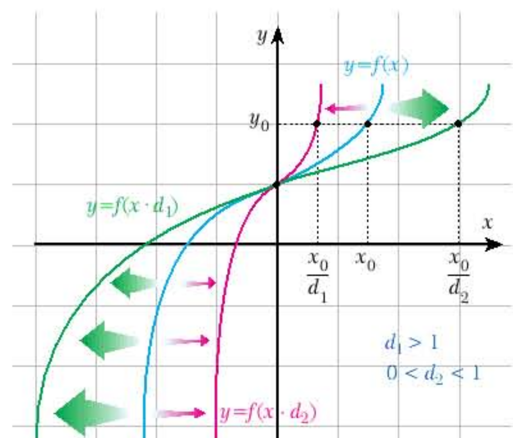
reflection in the y-axis



reflection in the x-axis



vertical shrink and stretch

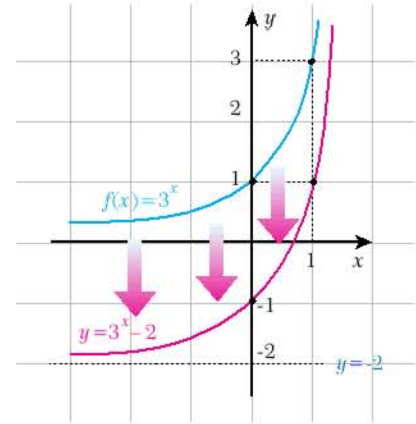


horizontal shrink and stretch

Now let us return to our original problem of sketching the graph of an exponential function. We can sketch the graph by considering $f(x)$ to be a^x in the transformations given above. For example, we can sketch the graph $y = 3^x - 2$ by noticing that it is in the form $y = f(x) - k$ where $f(x) = 3^x$ and $k = 2$. So the graph is the same as the graph of $f(x) = 3^x$ shifted 2 units downward.

We know that the points (0, 1) and (1, 3) are on the graph of $f(x) = 3^x$. These will move to (0, –1) and (1, 1), respectively, on $y = 3^x - 2$. For a vertical shift two units downward, we translate all the points on $f(x) = 3^x$ by subtracting 2 from their y-components (ordinates). Accordingly, we determine the following for $y = 3^x - 2$:

domain: $\mathbb{R}$, range: $(-2, \infty)$, asymptote: $y = -2$.

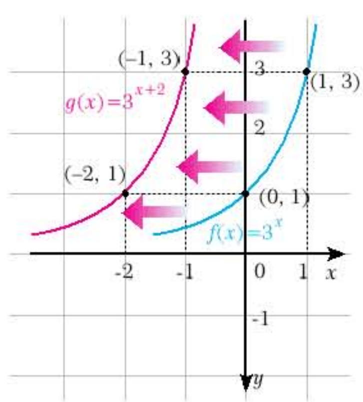


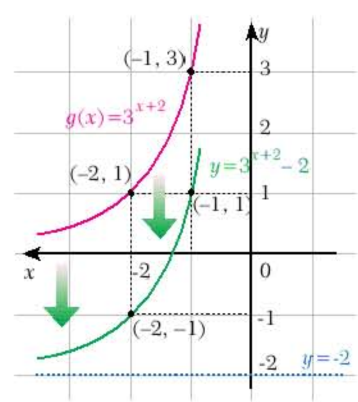
As we can see, the original asympote of $y = 3^x$ also shifts. In addition, notice that although the domain is still $\mathbb{R}$, the range of the function is affected by the vertical translation.

EXAMPLE 17 Sketch the graph of $y = 3^{x+2} - 2$ and determine its domain, range and asymptote.

Solution We will begin with the graph $y = 3^x$. Remember that when more than one transformation is applied to a function, we begin the graphical transformation by first considering the changes to the variable.

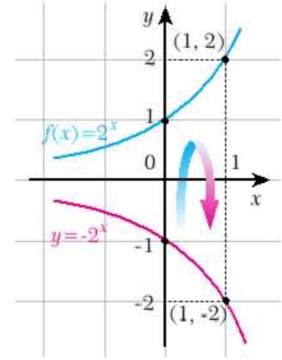
To obtain 3^{x+2} in terms of $f(x) = 3^x$, we can write $g(x) = 3^{x+2} = f(x + 2)$. So we shift the graph of $f(x) = 3^x$ two units to the left ($p = 2$). Then $y = 3^{x+2} - 2 = g(x) - 2$, so we shift the graph of $g(x)$ two units downward ($k = 2$).





As a result, we can see that $y = 3^{x+2} - 2$ has domain $\mathbb{R}$, range $(-2, \infty)$ and asymptote $y = -2$.

EXAMPLE 18 Sketch the graph of $y = -2^x$ and determine its domain, range and asymptote.



Solution We first draw the graph of $f(x) = 2^x$. By reflecting it in the x-axis we obtain the graph of $y = -2^x = -f(x)$. So $y = -2^x$ has domain $\mathbb{R}$, range $(-\infty, 0)$ and asymptote $y = 0$.

EXAMPLE 19 Sketch the graph of each function and determine its domain, range and asymptote.

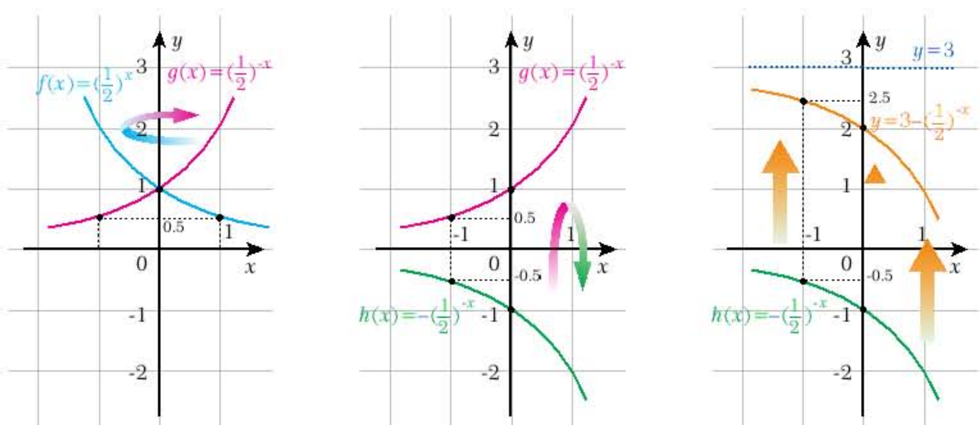
a. $f(x) = 3 - (\frac{1}{2})^{-x}$ b. $f(x) = 2^{2-x}$

Solution a. Step **1**: Sketch the graph of $f(x) = (\frac{1}{2})^x$.

Step **2**: Reflect the graph in the y-axis to obtain the graph of $g(x) = (\frac{1}{2})^{-x} = f(-x)$.

Step **3**: Reflect this new graph in the x-axis to obtain the graph of $h(x) = -(\frac{1}{2})^{-x} = -g(x)$.

Step **4**: Shift this graph three units upward, because $y = -(\frac{1}{2})^{-x} + 3 = h(x) + 3$. The result is the graph of $f(x) = 3 - (\frac{1}{2})^x$.



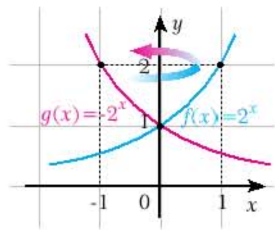
Steps 1 and 2 *Step 3* *Step 4*

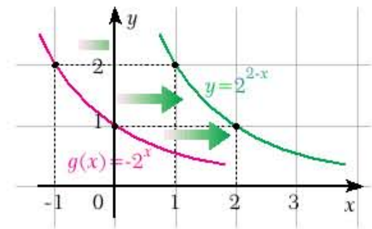
So $f(x) = 3 - (\frac{1}{2})^{-x}$ has domain $\mathbb{R}$, range $(-\infty, 3)$ and asymptote $y = 3$.

For $g(x) = 2^{-x}$,
$g(x + 2) = 2^{-(x+2)} \neq 2^{2-x}$.

b. We first sketch the graph of $f(x) = 2^{x}$ and reflect it in the y-axis to obtain $g(x) = 2^{-x} = f(-x)$. We then need to determine which value to use in $f(x)$ to obtain 2^{2-x}. We can use $y = g(x-2) = 2^{-(x-2)} = 2^{2-x}$. So we need to apply a horizontal shift with $p = 2$ to the graph of $g(x)$, i.e. we shift the graph of $g(x)$ two units to the right.

domain : $\mathbb{R}$
range : $(0, \infty)$
asymptote : $y = 0$





EXAMPLE 20 Sketch the graph of each function and determine its domain, range and asymptote.

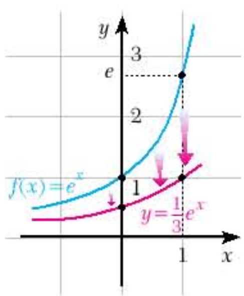
a. $f(x) = \frac{1}{3}e^{x}$

b. $y = 2^{\frac{1}{3}x}$

Solution

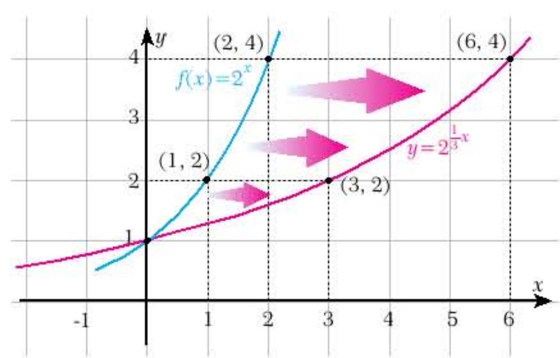
a. We first sketch the graph of $f(x) = e^{x}$, and then shrink it vertically by a factor of $\frac{1}{3}$, because $y = \frac{1}{3}e^{x} = \frac{1}{3}\cdot f(x)$ where $c = \frac{1}{3}$. Since $(0, 1)$ and $(1, e)$ are two points on $y = e^{x}$, the points $(0, \frac{1}{3})$ and $(1, \frac{e}{3})$ will be on $y = \frac{1}{3}e^{x}$.

So the function has domain $\mathbb{R}$, range $\mathbb{R}^{+}$ and asymptote $y = 0$.



b. We sketch the graph of $f(x) = 2^{x}$ and apply a horizontal stretch with a factor of $\frac{1}{d} = \frac{1}{\frac{1}{3}} = 3$ after identifying $y = 2^{\frac{1}{3}x} = f(\frac{1}{3}x)$ and $d = \frac{1}{3}$.

In other words, each point (x, y) on $f(x) = 2^{x}$ will move to $(3x, y)$.



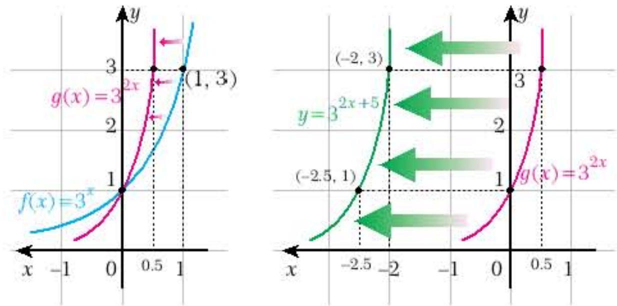
So the function has domain $\mathbb{R}$, range $\mathbb{R}^{+}$ and asymptote $y = 0$.

EXAMPLE 21 Sketch the graph of each function and determine its domain, range and asymptote.

a. $y = 3^{2x+5}$ b. $y = 2^{3-2x}$

Solution a. We sketch the graph of $f(x) = 3^x$ and write $g(x) = 3^{2x} = f(2x)$. We determine $d = 2$ and shrink the graph of $f(x)$ by a factor of $\frac{1}{d} = \frac{1}{2}$. Since

$$y = 3^{2x+5} = 3^{2\cdot(x+\frac{5}{2})} = g(x+\frac{5}{2}),$$



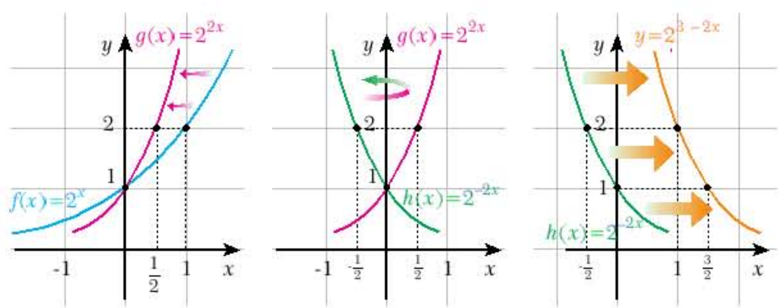
we shift the graph of $g(x)$ $\frac{5}{2}$ units to the left. So the function has domain $\mathbb{R}$, range $(0, \infty)$ and asymptote $y = 0$.

b. Starting with the graph of $f(x) = 2^x$ we follow the steps:

Step 1: Shrink the graph horizontally by a factor of $\frac{1}{2}$ to obtain the graph of $g(x) = 2^{2x} = f(\underbrace{2}_{d} \cdot x)$.

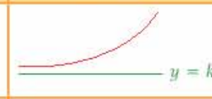
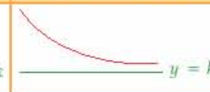
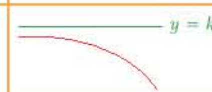
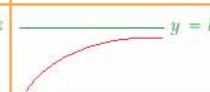
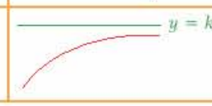
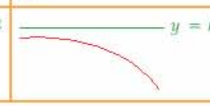
Step 2: Reflect this graph in the y-axis to obtain the graph of $h(x) = 2^{-2x} = g(-x)$.

Step 3: Shift this graph $\frac{3}{2}$ units to the right since $y = {}^{3-2x} = 2^{-2(x-\frac{3}{2})} = h(x-\frac{3}{2})$.



As we can see, applying step-by-step transformations to the basic exponential function $f(x) = a^x$ to graph a function $y = c \cdot a^{d(x+p)} + k$ is straightforward but it can take a long time. As an alternative approach, we can use a more basic procedure, as follows:

The two points $(0, 1)$ and $(1, a)$ on the graph of $f(x) = a^x$ will shift to $(-p, c + k)$ and $(\frac{1}{d} - p,\ c \cdot a + k)$ respectively for $y = c \cdot a^{d(x+p)} + k$. By plotting these two points and drawing the horizontal asymptote at $y = k$, we can obtain a rough graph of $y = c \cdot a^{d(x+p)} + k$. We can also use the following table:

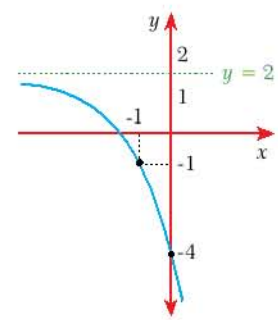
Graphs of functions of the form $y = c \cdot a^{d(x+p)} + k$					
c	d	$a > 1$	$0 < a < 1$	Domain	Range
+	+			$\mathbb{R}$	(k, ∞)
+	–			$\mathbb{R}$	(k, ∞)
–	+			$\mathbb{R}$	$(-\infty, k)$
–	–			$\mathbb{R}$	$(-\infty, k)$

EXAMPLE 22 Sketch each graph.

a. $y = -3 \cdot 2^{x+1} + 2$ b. $y = 2^{3-2x} - 1$

Solution a. By comparing the equation with the general form $y = c \cdot a^{d(x+p)} + k$, we identify $c = -3$, $d = 1$, $p = 1$ and $k = 2$. So the three points (x, y), $(0, 1)$ and $(1, 2)$ on the graph of $f(x) = 2^x$ will move as follows:

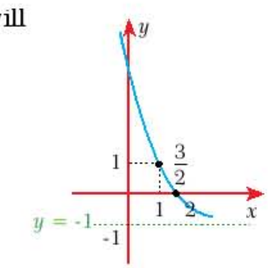
$f(x) = 2^x$		$y = -3 \cdot 2^{x+1} + 2$
(x, y)	$\rightarrow$	$(\frac{x}{d} - p,\ c \cdot y + k)$
$(0, 1)$	$\rightarrow$	$(\frac{0}{1} - 1, -3 \cdot 1 + 2) = (-1, -1)$
$(1, 2)$	$\rightarrow$	$(\frac{1}{1} - 1, -3 \cdot 2 + 2) = (0, -4)$.



We plot the points $(-1, -1)$ and $(0, -4)$, draw the horizontal asymptote $y = k = 2$ and then sketch the graph.

b. By rewriting the given equation as $y = 2^{3-2x} - 1 = 2^{-2(x - \frac{3}{2})} - 1$, we identify $c = 1$, $d = -2$, $p = -\frac{3}{2}$, and $k = -1$. So the points $(0, 1)$ and $(1, 2)$ on $f(x) = 2^x$ will move as follows:

$f(x)$		y
$(0, 1)$	$\rightarrow$	$(\frac{3}{2}, 0)$
$(1, 2)$	$\rightarrow$	$(1, 1)$.



Using $y = -1$ as the horizontal asymptote, we plot the points and sketch the graph.

EXERCISES 1.3

1. State the transformation(s) (i.e. the type of stretch, shrink or reflection) which should be applied to the graph $y = 2^x$ to obtain the graph of each function.

a. $y = 2^x + 1$

b. $y = 2^x - 3$

c. $y = 2^{x-2}$

d. $y = 2^{x+1}$

e. $y = -2^x$

f. $y = 2^{-x}$

g. $y = 2^{\frac{1}{2}x}$

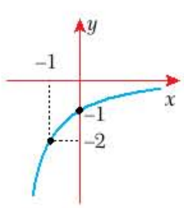
h. $y = 3 \cdot 2^x$

i. $y = -2^{x-2}$

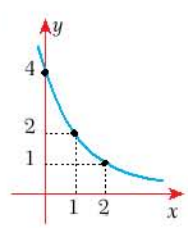
j. $y = 1 - 2^{x+1}$

2. Each graph shows a single transformation of the graph $f(x) = (\frac{1}{2})^x$. Write the equation of the new graph in each case.

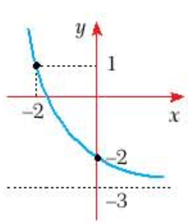
a.



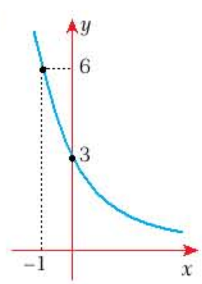
b.



c.



d.



3. Determine the domain D, range R and asymptote a for each function.

a. $y = 4 \cdot 2^x$

b. $y = 3 \cdot 2^{-x}$

c. $y = 2^{\frac{x}{2}+1}$

d. $y = -5^x$

e. $y = 2^x - 3$

f. $y = 4 + (\frac{1}{2})^x$

g. $y = 6 - 3^x$

h. $y = -(\frac{1}{5})^x$

i. $y = -e^{-x}$

4. Sketch each graph by plotting selected points.

a. $y = 2^x + 3$

b. $y = (\frac{1}{3})^x - 2$

c. $y = (\frac{1}{3})^{x-2}$

d. $y = 10^{x+2}$

e. $y = 10^{x-3} - 3$

f. $y = -4 + (\frac{1}{2})^{x-1}$

g. $y = -|5^x|$

h. $y = 2^{|x-1|}$

i. $y = 3 - (2 \cdot 3^{x-2})$

4 APPLICATIONS OF EXPONENTIAL FUNCTIONS

Exponential functions have many applications in the real world. Many of these applications are in physics, biology, electronic engineering, computer technology and economics. Exponential functions are also sometimes used in the social sciences: for example, the popularity of advertisements or fashions often decreases exponentially.

Any situation in which the rate of change of a quantity is proportional to the amount present can be modeled by an exponential function. Earlier in this book, we considered the example of a virus spreading through the population of your city. The rate at which the virus spreads is proportional to the number of people infected at a particular time. We can therefore model the spread of the virus with an exponential equation.

We have also seen that exponential functions are either increasing or decreasing. Real-life situations which can be modeled by increasing exponential functions are examples of exponential growth, while situations which can be modeled by decreasing exponential functions are examples of exponential decay.

Both exponential growth and decay can be modeled by an exponential function of the form

$$P = P_0 \cdot a^t.$$

The value P_0 represents the initial quantity that is growing or decaying. This could be an initial number of bacteria, the population of a city at a given time, or the original value of an investment.

a is the constant number and the base. It represents the rate of change. When $a > 1$ there is exponential growth, and when $0 < a < 1$ there is exponential decay. In situations where the growth or decay rate is given as a percentage, we can calculate a as follows:

1. for exponential growth, add the per cent increase to 100% and change the resulting percentage into a decimal.
2. for exponential decay, subtract the per cent decrease from 100% and change the resulting percentage into a decimal.

t is the time interval over which the growth or decay happens, in a given unit (days, years, decades, etc.).

A. EXPONENTIAL GROWTH

Exponential growth occurs when a quantity grows by a constant percentage in each fixed period of time. This means that the larger the quantity gets, the faster it grows. Population growth, the spread of a virus and the growth in the number of cells in a bacterial colony are all examples of exponential growth.

Here is a famous story which is based on exponential growth and which shows how its surprising characteristics have fascinated people through the ages.

The Legend of the Grain of Rice

There was once a clever courtier who presented a beautiful chess set to his king. In return, he asked only that the king place a grain of rice on the first square of the chessboard, and then double the number of grains he placed on each subsequent square. In other words, the king had to place one grain of rice on the first square, two grains on the second square, four grains on the third square and so on.

Since a chessboard has only 64 squares, this seemed to the king to be a modest request, so he called for his servants to bring the rice.

When the king's men started filling the squares with rice grains, the fun started. Though the eighth square needed only 128 grains, the 22nd square needed roughly 2 million rice grains. The 23rd square needed four million rice grains; the 24th 8 million. And there were still forty squares to go, with the last square needing 9,223,372,036,854,775,808 (roughly 9 million trillion) rice grains! In total, the king should have placed 18,446,744,073,709,551,615 grains on the board (adding up all the squares), which would be 549,755,830,887 tons of rice.

Obviously, the king could not keep his promise. Long before he reached the 64th square, every grain of rice in the kingdom had been used. Even today, the total world rice production would not be enough to meet the amount needed for the final square of the chessboard: the required amount would cover the entire surface of the Earth with rice fields two times over, oceans included. If one grain were counted out every second, it would take 584 billion years (i.e. nearly 130 times the age of the Earth) to count them all.

Square number (n)	1	2	3	4	5	...	n	...	64
Number of grains	1	2	4	8	16	...		...	
Formula	2^0	2^1	2^2	2^3	2^4	...	2^{n-1}	...	2^{63}

Obviously the king in the story had not studied mathematics, as he did not understand exponential growth. The secret to understanding the arithmetic is that the rate of growth (doubling for each square) applies to an ever-increasing amount of rice, so the number of grains added at each step goes up, even though the rate of growth is constant.

EXAMPLE 23 If the world population was 6.5 billion people in 2005 and if the population continues to grow by 1.5% annually, what will the approximate population be in 2010?

Solution We begin by identifying the values of P_0, a and t in the problem.

P_0 is the initial quantity. It corresponds to the population of the world in the year 2005: 6.5 billion.

Since the world population grows by a constant percentage (1.5%), we add 1.5% to 100% to obtain 101.5% and then convert 101.5% into decimal number. As a result, a is 1.015.

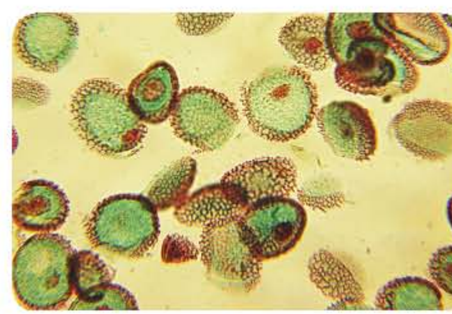
Since 2010 is five years after 2005, t is 5.

Substituting the values into the formula, we obtain

$$P = P_0 \cdot a^t = 6.5 \cdot (1.015)^5 \Leftrightarrow P = 6.5 \cdot 1.077 \Leftrightarrow P \approx 7.005.$$

So the world population in 2010 will be approximately 7 billion.

EXAMPLE 24 A colony of bacteria consists of 1000 bacteria. If the bacteria population triples every 30 minutes, what will the population be after 2 hours?



Solution Again we use $P = P_0 \cdot a^t$.

P_0 is the initial number of bacteria in the colony (1000).

Since the population triples for each period, we take the rate of growth, a, as 3.

To find t, we first convert two hours into minutes:

$$2 \cdot 60 = 120 \text{ minutes}$$

and divide this by 30 to identify the number of periods:

$$t = \frac{120}{30} = 4.$$

Now we substitute the values into the formula:

$$P = 1000 \cdot 3^4 = 81000.$$

So after two hours there will be 81000 bacteria in the colony.

B. EXPONENTIAL DECAY

Exponential decay occurs when a quantity decreases by a constant percentage in each fixed period of time. The decay of radioactive substances is an example of exponential decay.

All radioactive substances have a specific half-life, which is the time required for half of the particular substance to decay. For example, if we have 1000 grams of a substance with a half-life of 10 years, we can express the amount of the substance remaining after a given number of years as follows:

Archaeologists use a method called radiocarbon dating (also called carbon-14 dating) to find out how old archaeological discoveries are. Radiocarbon dating relies on the exponential decay of radioactive carbon-14 atoms in non-living things.

The carbon dioxide in the Earth's atmosphere always contains a fixed percentage of carbon-14 atoms. Plants absorb this carbon-14 with the carbon dioxide, and pass it on to other living creatures through the food chain. As a result, the ratio of normal carbon (carbon-12) to radiocarbon (carbon-14) in the atmosphere and in all living organisms is the same.

When an organism dies, the amount of carbon-12 it contains remains the same, but the carbon-14 decays exponentially with a half-life of about 5730 years. Scientists can therefore use the ratio of carbon-12 to carbon-14 in a non-living organism to find out how long ago it died.

Number of years	Amount of substance remaining
10	$1000 \cdot \frac{1}{2}$
20	$1000 \cdot \frac{1}{2} \cdot \frac{1}{2} = 1000 \cdot (\frac{1}{2})^2$
30	$1000 \cdot (\frac{1}{2})^2 \cdot \frac{1}{2} = 1000 \cdot (\frac{1}{2})^3$
t	$1000 \cdot (\frac{1}{2})^{\frac{t}{10}} = 1000 \cdot 2^{-\frac{t}{10}}$

More generally, we can say that the function $N(t) = N_0 \cdot 2^{-\frac{t}{h}}$ expresses the amount of a radioactive substance which remains after t years where h is the half-life (measured in the same unit as t) and N_0 is the original amount of the substance.

EXAMPLE 25 An isotope of the synthetic element Californium has a half-life of approximately 45 minutes. How long would it take for a sample of Californium to decay to 25% of its original mass?

Solution Let N_0 be the original mass of the sample and let t be the desired time (in minutes).

25% of N_0 means $0.25 \cdot N_0$. Using this value in the formula above gives us

$$0.25 \cdot N_0 = N_0 \cdot 2^{-\frac{t}{45}} \Leftrightarrow \frac{1}{4} = 2^{-\frac{t}{45}} \Leftrightarrow 2^{-2} = 2^{-\frac{t}{45}} \Leftrightarrow -2 = -\frac{t}{45} \Leftrightarrow t = 90.$$

So it would take approximately 90 minutes for the sample to decay.

C. COMPOUND INTEREST

If interest is **compounded**, its amount is based on an initial investment plus any previous interest.

If interest is compounded

- **annually** then $n = 1$,
- **quarterly** then $n = 4$,
- **monthly** then $n = 12$

in the compound interest formula.

Compound interest is important to anyone with money in a bank account which earns interest. It is also important in many financial calculations.

If an amount of money P, called the principal, is invested at an annual interest rate r (expressed as a decimal) compounded n times a year, then the amount A in the account at the end of t years is given by

$$\boxed{A = P\cdot(1+\frac{r}{n})^{nt}}.$$

This formula is often referred to as the compound interest formula.

EXAMPLE 26 You deposit \$200 in a bank that pays 7% interest compounded quarterly.

Find the amount in the account after 3 years, rounded to the nearest dollar.

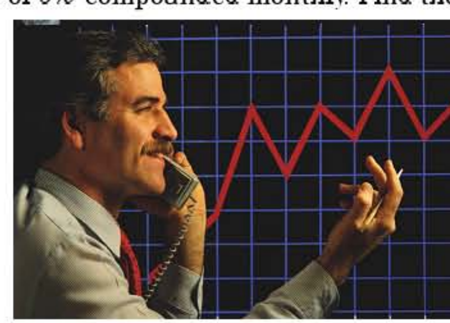
Solution After identifying $P = 200$, $r = 7\% = \frac{7}{100}$, $n = 4$ and $t = 3$, we apply the compound interest formula as follows:

$$A = P(1+\frac{r}{n})^{nt} = 200(1+\frac{0.07}{4})^{4\cdot 3} = 200(1+0.0175)^{12}$$

$$= 200(1.0175)^{12} \approx 200\cdot 1.23144 = 246.288.$$

Thus, after 3 years there will be approximately \$246 in the account.

EXAMPLE 27 \$1000 is invested in an account with an interest rate of 9% compounded monthly. Find the new principal after 5 years.



Solution Applying the compound interest formula with $r = 9\% = 0.09$, $n = 12$, $t = 5$ and $P = \$1000$, we find that the amount after five years is

$$A = 1000(1+\frac{0.09}{12})^{12\cdot 5} = 1000(1.0075)^{60} \approx \$1565.68.$$

Remark

If the interest is compounded continuously, the compound interest formula becomes

$$A = P \cdot e^{-rt}.$$

Check Yourself 5

1. The population of a small town grows by 5% each year. The population is now 4570. What will the population be in 10 years' time?
2. The number of bacteria in a certain colony doubles every 20 minutes. If the colony starts with a population of 800, how many bacteria will be present after 5 hours?
3. The half-life of radium is approximately 1600 years. How much of a 200-gram sample will be left after 100 years?
4. $2500 is invested at an interest rate of 15% compounded monthly. Find the new principal after 10 years.

Answers

1. 7444 2. $100 \cdot 2^{18}$ 3. approximately 192 g 4. $11,101

EXERCISES 1.4

A. Exponential Growth

1. The number of bacteria in a growing colony is given by the function $N(t) = 50 \cdot 2^t$, where t is the time in hours.

   a. How many bacteria were there initially?

   b. How many bacteria will there be after 30 minutes?

   c. How many bacteria will there be after 3 hours?

2. The population of a certain city was 450 000 in the year 2000 and is increasing with a relative growth rate of 8% per year.

   a. Estimate the population in 2008.

   b. Estimate the population in 2015.

3. A bacteria population doubles every ten minutes. If the population begins with one cell, how long will it take to grow to 1024 cells?

4. The number of bacteria in a colony is initially 39 and grows to 156 in 1 hour and 19 minutes. How many bacteria will be present 9 hours and 13 minutes after the initial measurement?

B. Exponential Decay

5. A new computer costs \$1000. It is estimated that each year it will lose 9% of the previous year's value. Find the value of the machine after

   a. the first year.

   b. 2 years.

   c. 12 years.

6. The half-life of a radioactive substance is 153 days. How many days will it take for 93.75% of a sample of the substance to decay?

7. The half-life of tritium is 12.4 years. How long will it take for 75% of a sample of tritium to decay?

8. An isotope of thorium, ^{327}Th, has a half-life of 18.4 days. How long will it take for 87.5% of the sample to decay?

C. Compound Interest

9. \$1000 is invested with an interest rate of 10%, compounded monthly. Find the value of the investment after

   a. 2 years. b. 4 years. c. 6 years.

10. Find the principal required to obtain \$600 interest after 2 years at 12% interest, compounded quarterly.

CHAPTER 1 SUMMARY

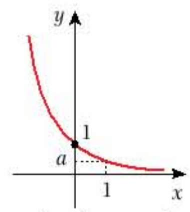
1. Exponents

- In the exponential expression a^p, a is called the base and p is called the exponent. We read this expression as 'the pth power of a' or 'p to the power a'. We can also read a^2 as 'a squared' and a^3 as 'a cubed'.
- For $n \in \mathbb{N}$ and $a \in \mathbb{R}$, $a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$.
- $a^1 = a$, $a^0 = 1$ where $a \neq 0$
- 0^0 is undefined.
- $a^{-n} = \dfrac{1}{a^n}$ where $a \in \mathbb{R}$, $a \neq 0$ and $n \in \mathbb{N}$
- Laws of Exponents
  1. $a^m \cdot a^n = a^{m+n}$
  2. $\dfrac{a^m}{a^n} = a^{m-n}$
  3. $(a^m)^n = a^{m \cdot n}$
  4. $(a \cdot b)^m = a^m \cdot b^m$
  5. $(\dfrac{a}{b})^m = \dfrac{a^m}{b^m}$
- Let a and b be real numbers, and let n be an integer greater than or equal to 2. Then an nth root of a is a number which, when raised to the power n, is equal to a. In other words, b is an nth root of a if and only if $b^n = a$.
- The principal nth root $\sqrt[n]{a}$ identifies
  a. the positive root of a when n is even and a is positive.
  b. the unique root which has the same sign as a when n is odd.

  When n is even and a is negative, the principal nth root of a is undefined in the set of real numbers.
- Properties of n^{th} Roots
  1. $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
  2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$
  3. $\sqrt[n]{\sqrt[m]{a}} = \sqrt[m \cdot n]{a}$
- $\sqrt[n]{a} = a^{\frac{1}{n}}$
- If $b = a^{\frac{1}{n}}$ then $b^n = a$.
- $a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$

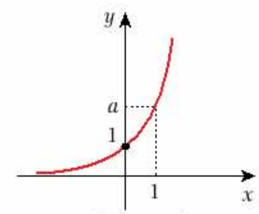
2. Exponential Functions

- A function of the form $f(x) = a^x$ for $a>0$ and $a \neq 0$ is called a basic exponential function.
- Basic exponential functions have a graph similar to one of the following:



when $0 < a < 1$

or



when $a > 1$

- Properties of the Basic Exponential Function $f(x) = a^x$
  1. The domain is the set of real numbers.
  2. The range is the set of positive real numbers.
  3. The x-axis ($y = 0$) is a horizontal asymptote.
  4. There is no x-intercept.
  5. The y-intercept is $(0, 1)$.
  6. $(0, 1)$ and $(1, a)$ are two points on $f(x) = a^x$.
  7. The function is strictly increasing when $a > 1$ and strictly decreasing when $0 < a < 1$
  8. The function is bijective, and therefore has an inverse (known as a logarithmic function).

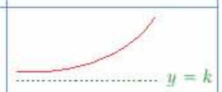
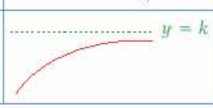
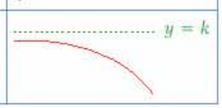
3. Simple Variations of Exponential Functions

- A function of the form $f(x) = c \cdot a^{d(x+p)} + k$ where a, c, d, p and k are real numbers with $a > 0$ and $a \neq 1$ is called an exponential function with base a.
- The graph of an exponential function can be obtained by transforming the graph of a basic exponential function a^x.
- For a function of the form $f(x) = c \cdot a^{d(x+p)} + k$,
  - k represents a vertical shift,
  - p represents a horizontal shift,
  - c represents a vertical stretch or shrink,
  - d represents a horizontal stretch or shrink

  of the graph of the basic exponential function $y = a^x$.
- Any point (x, y) on $y = a^x$ moves to

  $(\dfrac{x}{d} - p,\ c \cdot y + k)$ for $f(x) = c \cdot a^{d(x+p)} + k$.

- Properties of the General Exponential Function $f(x) = c \cdot a^{d(x+p)} + k$
  1. The domain is $\mathbb{R}$.
  2. $y = k$ is a horizontal asymptote of the graph.
  3. $(-p, c+k)$ and $(\frac{1}{d} - p, c \cdot a + k)$ are two points on the graph.
- The following table shows how the values of c and d affect the behavior of $f(x) = c \cdot a^{d(x+p)} + k$:

c	d	$a > 1$	$0 < a < 1$
+	+		
+	−		
−	+		
−	−		

4. Applications of Exponential Functions

- Any real-life situation in which the rate of change of a quantity is proportional to the amount present can be modeled by an exponential function.
- Real-life situations which can be modeled by increasing exponential functions are examples of exponential growth, while situations which can be modeled by decreasing exponential functions are examples of exponential decay.
- Population growth is an example of exponential growth. The decay of a radioactive substance is an example of exponential decay.
- Both exponential growth and decay can be modeled by an exponential function of the form

$$P = P_0 \cdot a^t.$$

- When $a > 1$, there is exponential growth, and when $0 < a < 1$ there is exponential decay.
- Compound interest: If an amount of money P (called the principal) is invested at an interest rate r (expressed as a decimal) compounded n times a year then the amount A in the account at the end of t years is given by

$$A = P \cdot (1 + \frac{r}{n})^{nt}.$$

Concept Check

1. What is an exponential expression?
2. What is the difference between a power and an exponent?
3. What happens if the base of an exponential expression is equal to 1?
4. Give an example of an exponential expression which is not a real number.
5. How can we calculate an irrational power of a real number?
6. Are $f(x) = x^n$ and $g(x) = x^x$ exponential functions?
7. How does the base a affect the graph of a basic exponential function?
8. What is an asymptote?
9. Basic exponential functions are monotone. Explain what this means.
10. When does a basic exponential function have negative values?
11. When does a basic exponential function have values greater than 1?
12. What is the relation between basic exponential functions ($f(x) = a^x$) and general exponential functions

    ($f(x) = c \cdot a^{d(x+p)} + k$)?
13. What are the common properties of basic exponential functions? What are the properties of general exponential functions?
14. How does the value of c affect the behavior of the exponential function $f(x) = c \cdot a^{d(x+p)} + k$? Explain the effect of the other constants.
15. To which point will (x_0, y_0) on $f(x) = a^x$ move for

    $y = 2 \cdot a^{5(x-1)} - 3$?
16. When is the growth or decay of a quantity considered to be exponential?
17. Give three examples of situations which can be modeled by exponential functions.

CHAPTER REVIEW TEST 1A

1. An exponential function is of the form $f(x) = a^x$. Given $f(3) = \frac{1}{8}$, calculate $f(-2)$.

A) $\frac{1}{8}$ B) $\frac{1}{4}$ C) 2 D) 4 E) 8

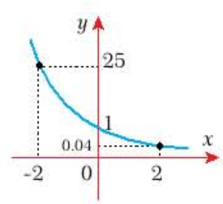
2. Simplify $(-a)^6 \cdot (-a^4) \cdot (-a)^5 \cdot (\frac{1}{a})^{10}$.

A) a^5 B) $-a^5$ C) a^{25} D) $-a^{25}$ E) a^{15}

3. $a^x = 2$ is given. What is $\dfrac{a^{2x} - a^{-2x}}{a^x}$?

A) $\frac{1}{2}$ B) $\frac{3}{4}$ C) $\frac{7}{4}$ D) $\frac{7}{8}$ E) $\frac{15}{8}$

4. Which graph is shown in the figure opposite?



A) $y = 5^x$ B) $y = 2^x$ C) $y = 2^{-x}$ D) $y = 4^{-x}$ E) $y = 5^{-x}$

5. $f(x) = 8 \cdot 7^x$ is given. Calculate $f^{-1}(8)$.

A) 56 B) 8 C) 7 D) 1 E) 0

6. Which statement is <u>false</u>?

A) $4^{-3} + 4^{-3} + 4^{-3} + 4^{-3} = 4^{-2}$

B) $4^{-6} = 2^{-12}$

C) $(4^{-2})^3 = 4^{-5}$

D) $4^{-1} + 4^{-1} = 2^{-1}$

E) $2 \cdot 4^{-3} = 2^{-5}$

7. Solve $8^{41} = x \cdot 2^{125}$ for x.

A) $\frac{1}{8}$ B) $\frac{1}{4}$ C) $\frac{1}{2}$ D) 2 E) 4

8. Which of the following is not an exponential function for $x \in \mathbb{R}$?

A) $f(x) = 2^{2x}$ B) $f(x) = 3^{-2x}$ C) $f(x) = -7^{x/3}$ D) $f(x) = (-2)^x$ E) $f(x) = 11^{-x/4}$

9. Calculate $\dfrac{2^9 \cdot 5^7}{10^6}$.

A) 400 B) 40 C) 4 D) 0.4 E) 0.04

10. Which of the following is equivalent to $\sqrt{5 + 2\sqrt{6}}$?

A) $7 + \sqrt{6}$ B) $\sqrt{3} + \sqrt{2}$ C) $1 + \sqrt{6}$ D) $\sqrt{3} + 1$ E) $\sqrt{2} + 2$

11. Which of the following is greater than 1?

A) $(\frac{1}{\pi})^2$ B) $(0.\bar{3})^{-2}$ C) $(\frac{10}{3})^{-\frac{10}{3}}$

D) $(-\frac{1}{4})^4$ E) $(\frac{1}{\sqrt{6}})^{0.6}$

12. Calculate $\frac{4-\sqrt{12}}{\sqrt{7-4\sqrt{3}}}$.

A) 2 B) 5 C) 7 D) 4 E) 6

13. Calculate $\frac{3^{103}-3^{102}}{9^{52}}$.

A) $\frac{1}{3}$ B) $\frac{1}{9}$ C) $\frac{2}{3}$ D) $\frac{2}{9}$ E) 3

14. What is half of 4^{20}?

A) 2^{20} B) 4^{10} C) 2^{10} D) 2^{39} E) 4^{19}

15. Evaluate $\frac{5^{1.1}}{125^{-0.3}}+\frac{9^{0.4}}{9^{-0.1}}$.

A) 25.01 B) 23 C) 25.9 D) 28 E) 34

16. $\frac{3\cdot 2^{2n-1}-6\cdot 2^{2n-3}}{3\cdot 2^{2n+1}}=2^n$ is given. Find n.

A) 1 B) –2 C) 2 D) 3 E) –3

17. How many digits are there in the number $5^{10}\cdot 4^4\cdot 6^3$?

A) 10 B) 11 C) 12 D) 13 E) 14

18. Evaluate $\sqrt{(\sqrt{2}-5)^2}+\sqrt{(-5)^2}+\sqrt{2}$.

A) $2\sqrt{2}$ B) 0 C) $2\sqrt{2}-10$ D) 10 E) –10

19. Which statement is not necessarily true for the exponential function $f(x) = a^x$?

A) $f(x)$ is injective B) $f(x)$ is bijective

C) $f(x)$ is increasing D) $f(x)$ is surjective

E) $f(x)$ is positive

20. Given that $a=\sqrt[6]{\frac{4}{\sqrt{2}}}$, which of the following is an integer?

A) a^6 B) a^2 C) a^3 D) a^4 E) a^5

CHAPTER REVIEW TEST 1B

1. What is half of 4^{-19}?

A) 2^{-39} B) 2^{-38} C) 2^{-37} D) 2^{-19} E) 2^{-18}

2. Evaluate $\dfrac{x^{y}-y^{-x}}{x^{-y}-y^{x}}$ for $x, y \in \mathbb{R}$.

A) $(\dfrac{x}{y})^{x}$ B) $(\dfrac{x}{y})^{-x}$ C) $-(\dfrac{x}{y})^{x}$ D) $-\dfrac{x^{y}}{y^{x}}$ E) $-\dfrac{y^{x}}{x^{y}}$

3. $A = 3^{x} - 3^{-x}$ and $B = 3^{x} + 3^{-x}$ are given. Which of the following shows the relation between A and B?

A) $A^{2} - B^{2} = 4$ B) $A^{2} - B = 2$

C) $A \cdot B = 4$ D) $B^{2} - A^{2} = 4$

E) $A^{2} + B^{2} = 4$

4. Evaluate $\dfrac{3^{2x+1}-(2 \cdot 3^{2x-1})+(2 \cdot 3^{2x})}{(2 \cdot 9^{x})+(4 \cdot 9^{x-1})-3^{2x}}$.

A) 5 B) 4 C) 3 D) 2 E) 1

5. Evaluate $\dfrac{27^{\frac{2}{3}}}{16^{\frac{1}{2}}} \cdot 10^{2}$.

A) 45 B) 75 C) 100 D) 225 E) 450

6. Given $5^{a} - 3^{a} = k$, evaluate $\dfrac{10^{a}+6^{a}}{50^{a}-18^{a}}$ in terms of k.

A) k^{2} B) k C) $\dfrac{1}{k}$ D) $\dfrac{1}{k^{2}}$ E) $\dfrac{1}{k^{3}}$

7. Which statement is <u>false</u>?

A) $4^{15} + 4^{15} = 2^{31}$ B) $2^{15} \cdot 2^{15} = 4^{15}$

C) $2^{15} + 2^{15} = 4^{15}$ D) $2^{-15} > 0$

E) $4^{15} \cdot 2^{15} = 2^{45}$

8. Given $11^{443} = a$, write $\dfrac{11^{888}+11^{886}-122}{11^{443}-1}$ in terms of a.

A) $122a$ B) $122a^{2}$ C) $122(a-1)$

D) $122(a+1)$ E) $a+1$

9. Evaluate $\dfrac{2^{95}+2^{94}+2^{90}}{2^{92}+2^{91}+2^{87}}$.

A) $\dfrac{1}{8}$ B) $\dfrac{1}{4}$ C) 4 D) 8 E) 16

10. If $x \cdot y \cdot z \neq 0$, which of the following cannot be zero?

A) $x^{2} + y^{2} + z$ B) $x^{2} - y^{4} + z^{6}$

C) $x + y + z$ D) $(x + y + z)^{2}$

E) $x^{2} + (y + z)^{2}$

11. Evaluate $\dfrac{0.05 \cdot 10^5 + 3000}{0.005 \cdot 10^4 - 0.01 \cdot 10^3}$.

A) 100 B) 200 C) 400 D) 2000 E) 4000

12. $a = 8^{100}$, $b = 243^{40}$ and $c = (0.008)^{-50}$ are given. Which statement is true?

A) $c < a < b$
B) $c < b < a$
C) $a < b < c$
D) $a < c < b$
E) $b < c < a$

13. Evaluate $\dfrac{1}{7^{m-2}-1} + \dfrac{1}{7^{2-m}-1}$.

A) 1 B) 7^m C) $7^m - 1$ D) 7^2 E) -1

14. $(0.08 \cdot 0.2) = (a + 0.6) \cdot 10^b$ is given where $a, b \in \mathbb{Z}$. What is a possible value of $a + b$?

A) -3 B) -2 C) -1 D) 1 E) 2

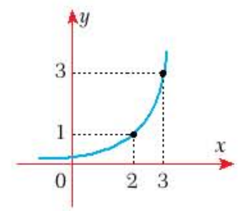
15. Simplify $\left(\dfrac{a^x}{a^y}\right)^{x-y} \cdot \left(\dfrac{a^y}{a^x}\right)^{x-y}$.

A) a^y B) a C) a^x D) 1 E) a^{x-y}

16. How many consecutive zeros are there at the end of $(75 \cdot 12 \cdot 5 \cdot 8 \cdot 40)^4$?

A) 13 B) 14 C) 15 D) 16 E) 17

17. Which function has the graph shown opposite?



A) $f(x) = 3^x - 2$
B) $f(x) = 2^{x-3}$
C) $f(x) = 3^{x-2}$
D) $f(x) = 3^x - 3$
E) $f(x) = 3^{x-1}$

18. $\dfrac{\sqrt[3]{2^6} \cdot \sqrt{32^{-32}}}{\sqrt{32^{-29}} \cdot \sqrt{128}} = 8^n$ is given. Find n.

A) -3 B) 5 C) 7 D) -4 E) 6

19. Evaluate $(\sqrt{5} - 1) \cdot \left(1 + \dfrac{2}{\sqrt{5}} + \dfrac{3\sqrt{5}}{5}\right)$.

A) 3 B) 4 C) 5 D) 6 E) 7

20. Evaluate $\dfrac{\sqrt[3]{128} - \sqrt[3]{16}}{\sqrt[3]{250} - \sqrt[3]{54}}$.

A) $\dfrac{2}{5}$ B) $\dfrac{2}{3}$ C) 1 D) $\dfrac{3}{2}$ E) $\dfrac{5}{2}$

Chapter 2

LOGARITHMS AND LOGARITHMIC FUNCTIONS

1 LOGARITHMS

Consider the relation $a^x = N$. Imagine that we are asked to find one of the three numbers a, x or N given the other two numbers. Three examples of this are shown below.

Case	Solution	Method
$2^5 = p$	32	raise 2 to the 5th power
$p^3 = 27$	3	take the 3rd root of 27
$3^p = 5$	?	?

We can see that we cannot solve the last example with the algebra we have studied so far. We need to introduce a new concept: a logarithm.

A. BASIC CONCEPT

The logarithm of a number N to a base a is the power to which a must be raised in order to obtain N. We write this as $\log_a N$. In other words,

$$a^{\log_a N} = N \text{ where } a^x = N \text{ and } x = \log_a N.$$

This equation is called the fundamental identity of logarithms. In this equation, the base of the logarithm (a) is always positive and different from 1, and the number whose logarithm is taken (N) is positive. In other words, negative numbers and zero do not have logarithms.

Definition logarithm, argument, base, exponential form, logarithmic form

For $a > 0$, $a \neq 1$ and $x > 0$, the real number y which is defined by

$$y = \log_a x \Leftrightarrow a^y = x$$

is called the logarithm of x to the base a. In this notation, x is called the argument of the logarithm.

We say that the equation $a^y = x$ is in exponential form and $\log_a x = y$ is the same equation in logarithmic form.

EXAMPLE 1 Write the equalities in logarithmic form.

a. $2^3 = 8$ b. $5^0 = 1$ c. $3^{-2} = \dfrac{1}{9}$

Solution By the definition of a logarithm, $a^y = x \Leftrightarrow y = \log_a x$. Therefore,

a. $2^3 = 8 \Leftrightarrow 3 = \log_2 8$. b. $5^0 = 1 \Leftrightarrow 0 = \log_5 1$. c. $3^{-2} = \dfrac{1}{9} \Leftrightarrow -2 = \log_3 \dfrac{1}{9}$.

EXAMPLE 2 Write the equalities in exponential form.

a. $\log_{10} 100 = 2$ b. $\log_3 \frac{1}{27} = -3$ c. $\log_2 1 = 0$

Solution Again we use the definition $\log_a x = y \Leftrightarrow x = a^y$.

a. $\log_{10} 100 = 2 \Leftrightarrow 100 = 10^2$ b. $\log_3 \frac{1}{27} = -3 \Leftrightarrow \frac{1}{27} = 3^{-3}$

c. $\log_2 1 = 0 \Leftrightarrow 1 = 2^0$

EXAMPLE 3 Solve each equation for x.

a. $\log_x 27 = 3$ b. $\log_4 x = \frac{1}{2}$ c. $\log_4 16 = x$

Solution a. $\log_x 27 = 3 \Leftrightarrow x^3 = 27 \Leftrightarrow x = 3$ b. $\log_4 x = \frac{1}{2} \Leftrightarrow 4^{\frac{1}{2}} = x \Leftrightarrow x = 2$

c. $\log_4 16 = x \Leftrightarrow 4^x = 16 \Leftrightarrow 4^x = 4^2 \Leftrightarrow x = 2$

EXAMPLE 4 Calculate the logarithms.

a. $\log_2 4$ b. $\log_3 \frac{1}{9}$ c. $\log_2(\log_3 9)$

Solution a. Let $\log_2 4 = y$.

Then $\log_2 4 = y \Leftrightarrow 2^y = 4 \Leftrightarrow 2^y = 2^2 \Leftrightarrow y = 2$, so $\log_2 4 = 2$.

> Remember:
> $a^x = a^y \Leftrightarrow x = y$
> by the bijective property of exponential functions.

b. Similarly, $\log_3 \frac{1}{9} = y \Leftrightarrow 3^y = \frac{1}{9} \Leftrightarrow 3^y = 3^{-2} \Leftrightarrow y = -2$.

c. Let $\log_3 9 = m$. Then $3^m = 9 \Leftrightarrow 3^m = 3^2 \Leftrightarrow m = 2$. So we need to calculate $\log_2 2$. Starting with $\log_2 2 = n$, we get $2^n = 2$ which gives us $n = 1$. Thus, $\log_2(\log_3 9) = 1$.

EXAMPLE 5 Calculate the logarithms.

a. $\log_3 \frac{1}{3}$ b. $\log_{\frac{1}{3}} \sqrt[3]{81}$ c. $\log_a \sqrt[3]{a\sqrt{a}}$ d. $\log_3(\log_2(\log_9 81))$

Solution a. By the definition of a logarithm, we can write $\log_3 \frac{1}{3} = y \Leftrightarrow 3^y = \frac{1}{3} \Leftrightarrow 3^y = 3^{-1} \Leftrightarrow y = -1$.

So $\log_3 \frac{1}{3} = -1$.

b. $\log_{\frac{1}{3}} \sqrt[3]{81} = y \Leftrightarrow (\frac{1}{3})^y = (81)^{\frac{1}{3}} \Leftrightarrow 3^{-y} = (3^4)^{\frac{1}{3}} \Leftrightarrow 3^{-y} = 3^{\frac{4}{3}} \Leftrightarrow y = -\frac{4}{3}$

c. $\log_a \sqrt[3]{a\sqrt{a}} = y \Leftrightarrow a^y = (a\sqrt{a})^{\frac{1}{3}} \Leftrightarrow a^y = (a \cdot a^{\frac{1}{2}})^{\frac{1}{3}} \Leftrightarrow a^y = (a^{\frac{3}{2}})^{\frac{1}{3}} \Leftrightarrow a^y = a^{\frac{1}{2}} \Leftrightarrow y = \frac{1}{2}.$

So $\log_a \sqrt[3]{a\sqrt{a}} = \frac{1}{2}.$

d. Starting from the innermost logarithm, we have

$\log_9 81 = x \Leftrightarrow 9^x = 81 \Leftrightarrow 9^x = 9^2 \Leftrightarrow x = 2.$

So we have to calculate $\log_3 (\log_2 2)$, and $\log_2 2 = y \Leftrightarrow 2^y = 2 \Leftrightarrow y = 1.$

So the given expression becomes $\log_3 1$, which is equal to zero:

$\log_3 1 = z \Leftrightarrow 3^z = 1 \Leftrightarrow z = 0.$ In conclusion, $\log_3(\log_2(\log_9 81)) = 0.$

Notice that in these examples we were able to find the desired logarithm by writing the argument as a rational power of the base. This is not always possible, however: many logarithms (for example: $\log_2 3$ and $\log_3 5$) are irrational, and cannot be calculated in this way.

EXAMPLE 6 Evaluate the expressions.

a. $2^{\log_2 8}$ b. $25^{\log_5 3}$ c. $3^{3 \cdot \log_3 2}$

Solution a. By the fundamental identity of logarithms, $a^{\log_a N} = N$ and so $2^{\log_2 8}$ will be equal to 8. However, let us try to evaluate the expression in a different way:

Let $\log_2 8 = t$. Then we have to calculate 2^t.

By definition we have $\log_2 8 = t \Leftrightarrow 2^t = 8$. So $2^{\log_2 8} = 8$.

$(a^m)^n = (a^n)^m = a^{m \cdot n}$

b. Let $\log_5 3 = t$. Then we have to calculate 25^t. By definition, $\log_5 3 = t \Leftrightarrow 5^t = 3$. So $25^t = (5^2)^t = (5^t)^2 = (3)^2 = 9$, i.e. $25^{\log_5 3} = 9$.

c. In a similar way, let $\log_3 2 = t$ and let us calculate 3^{3t}. By definition, $\log_3 2 = t \Leftrightarrow 3^t = 2$, i.e. $3^{3t} = (3^t)^3 = 2^3 = 8$. So $2^{3 \cdot \log_3 2} = 8$.

Check Yourself 1

1. Write the equalities in logarithmic form.

a. $2^4 = 16$ b. $10^3 = 1000$ c. $3^0 = 1$ d. $125^{\frac{1}{3}} = 5$ e. $3^{-3} = \frac{1}{27}$ f. $(2\sqrt{2})^{-\frac{2}{3}} = \frac{1}{2}$

2. Write the equalities in exponential form.

a. $\log_{10} 0.01 = -2$ b. $\log_{\frac{1}{2}} \frac{1}{16} = 4$ c. $\log_{10} 10000 = 4$

d. $\log_3 \frac{1}{81} = -4$ e. $\log_2 32 = 5$ f. $\log_{\frac{1}{5}} 125 = -3$

3. State whether each statement is true or false.

a. $\log_3 729 = 6$ b. $\log_{\frac{1}{2}} \sqrt[3]{4} = -\frac{2}{3}$ c. $\log_{10} \frac{1}{10\sqrt{10}} = -\frac{3}{2}$

d. $\log_{3\sqrt{3}} \frac{1}{3} = -\frac{3}{2}$ e. $\log_a \sqrt{a\sqrt{a\sqrt{a}}} = \frac{7}{8} \quad (a > 0,\ a \neq 1)$

4. Determine the logarithms of each set of numbers to the given base.

a. $27,\ 1,\ \frac{1}{9},\ \frac{1}{\sqrt[3]{3}},\ \sqrt[3]{9},\ \frac{9}{\sqrt[4]{3}}$ to base 3 b. $2,\ 4,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{32},\ 32,\ -64$ to base $\frac{1}{2}$

5. Solve for x.

a. $\log_x 4 = 2$ b. $\log_4 x = -\frac{1}{2}$ c. $\log_{25} 125 = x$

6. Calculate the logarithms.

a. $\log_5 25$ b. $\log_6 1296$ c. $\log_{49} \frac{1}{7}$ d. $\log_3(\log_2 (\log_2 256))$

7. Evaluate the expressions.

a. $3^{-\log_3 4}$ b. $(2^{\log_2 5})^2$ c. $25^{-\log_5 10}$ d. $49^{\frac{1}{2}-\log_7 \frac{1}{4}}$ e. $2^{2\log_2 5+\log_2 3}$

Answers

1. a. $\log_2 16 = 4$ b. $\log_{10} 1000 = 3$ c. $\log_3 1 = 0$

d. $\log_{125} 5 = \frac{1}{3}$ e. $\log_3 \frac{1}{27} = -3$ f. $\log_{2\sqrt{2}} \frac{1}{2} = -\frac{2}{3}$

2. a. $10^{-2} = 0.01$ b. $(\frac{1}{2})^4 = \frac{1}{16}$ c. $10^4 = 10000$

d. $3^{-4} = \frac{1}{81}$ e. $2^5 = 32$ f. $(\frac{1}{5})^{-3} = 125$

3. a. true b. true c. true d. false e. true

4. a. $3,\ 0,\ -2,\ -\frac{1}{3},\ \frac{2}{3},\ \frac{7}{4}$ b. $-1,\ -2,\ 0,\ 1,\ 2,\ 5,\ -5$, undefined

5. a. 2 b. $\frac{1}{2}$ c. $\frac{3}{2}$

6. a. 2 b. 4 c. $-\frac{1}{2}$ d. 1

7. a. $\frac{1}{4}$ b. 25 c. $\frac{1}{100}$ d. $\frac{1}{4}$ e. 75

B. TYPES OF LOGARITHM

1. Common Logarithms

Our counting system is based on the number 10. For this reason, a lot of logarithmic work uses the base 10. Logarithms to the base 10 are called common logarithms. We often write $\log x$ or $\lg x$ to mean $\log_{10} x$. In this module, we will use $\log x$ to mean $\log_{10} x$.

Common logarithms are widely used in computation. Mathematicians have compiled extensive and highly accurate tables of common logarithms for use in these calculations. These tables and their use will be discussed later in this module.

2. Natural Logarithms

$e = 2.71828...$

Logarithms to the base e are called natural logarithms or Euler logarithms. We often write $\ln x$ to mean the natural logarithm $\log_e x$.

Natural logarithms are widely used in mathematical analysis in the study of limits, derivatives and integrals.

C. PROPERTIES OF LOGARITHMS

Property 1

If the argument and the base of a logarithm are equal, the logarithm is equal to 1. Conversely, if the logarithm is 1 then the argument and the base are equal:

$$\boxed{a = b \Leftrightarrow \log_a b = 1} \quad (a > 0,\ a \neq 1).$$

Proof

By the fundamental identity of logarithms we have $a^{\log_a N} = N$. Setting $N = a$ gives us $a^{\log_a a} = a = a^1$, which gives us $\log_a a = 1$.

For example, $\log_3 3 = 1$, $\log 10 = 1$, $\ln e = 1$ and $\log_{\frac{1}{2}} \frac{1}{2} = 1$.

Property 2

The logarithm of 1 to any base is zero:

$$\boxed{\log_a 1 = 0}.$$

Proof

$a^{\log_a 1} = 1 = a^0$. So $a^{\log_a 1} = a^0$, which gives us $\log_a 1 = 0$.

For example, $\log_3 1 = 0$, $\log_{\frac{1}{2}} 1 = 0$ and $\log_\pi 1 = 0$.

Property 3

The logarithm of the product of two or more positive numbers to a given base is equal to the sum of the logarithms of the numbers to that base:

$$\boxed{\log_a(x \cdot y) = \log_a x + \log_a y} \qquad (x, y > 0).$$

Proof $a^{\log_a(x \cdot y)} = x \cdot y$. Substituting $x = a^{\log_a x}$ and $y = a^{\log_a y}$ gives us

$a^{\log_a(x \cdot y)} = a^{\log_a x} \cdot a^{\log_a y} = a^{\log_a x + \log_a y}$.

Comparing the exponents of the expressions on both sides gives us the required equation:

$$\log_a(x \cdot y) = \log_a x + \log_a y.$$

Be careful!
$\log_a(x + y) \neq \log_a x + \log_a y$

For example,

$$\log_2 6 = \log_2(2 \cdot 3) = \log_2 2 + \log_2 3 = 1 + \log_2 3$$

$$\log_3 30 = \log_3(3 \cdot 10) = \log_3 3 + \log_3 10 = 1 + \log_3 10$$

$$\log_3 30 = \log_3(6 \cdot 5) = \log_3 6 + \log_3 5$$

$$\log_2 5 + \log_2 3 = \log_2(5 \cdot 3) = \log_2 15.$$

Notice that we can generalize this property as follows:

$$\boxed{\log_a(x_1 \cdot x_2 \cdot x_3 \cdot ... \cdot x_k) = \log_a x_1 + \log_a x_2 + ... + \log_a x_k} \; (x_1, x_2, x_3, \ldots, x_k > 0).$$

For example, we can write

$$\log_2 30 = \log_2(2 \cdot 3 \cdot 5) = \log_2 2 + \log_2 3 + \log_2 5 = 1 + \log_2 3 + \log_2 5.$$

EXAMPLE 7 Calculate $\log_4 2 + \log_4 8$.

Solution $\log_4 2 + \log_4 8 = \log_4(2 \cdot 8) = \log_4(4 \cdot 4) = \log_4 4 + \log_4 4 = 1 + 1 = 2$

EXAMPLE 8 Calculate $\log_2 3 + \log_2 5 + \log_2 \dfrac{1}{15}$.

Solution $\log_2 3 + \log_2 5 + \log_2 \dfrac{1}{15} = \log_2(3 \cdot 5 \cdot \dfrac{1}{15}) = \log_2 1 = 0$

Property 4 The logarithm of the power of a positive number is equal to the product of the power and the logarithm of the number.

$$\boxed{\log_a(x^m) = m \cdot \log_a x} \quad (m \in \mathbb{R},\ x > 0).$$

Be careful!
$(\log_a x)^m \neq m \cdot \log_a x$

Proof $x^m = a^{\log_a(x^m)}$. After substituting $x = a^{\log_a x}$ on the left side, we get $(a^{\log_a x})^m = a^{\log_a(x^m)}$, which gives us $a^{m \cdot \log_a x} = a^{\log_a(x^m)}$. Since the bases are the same on both sides, we can conclude $m \cdot \log_a x = \log_a(x^m)$.

For example,

$(a^m)^n = a^{m \cdot n}$

$$\log_2 8 = \log_2(2^3) = 3 \cdot \log_2 2 = 3 \cdot 1 = 3$$
$$\log_3 \frac{1}{243} = \log_3 \frac{1}{3^5} = \log_3(3^{-5}) = -5 \cdot \log_3 3 = -5 \cdot 1 = -5$$
$$\log_2 \sqrt{125} = \log_2 \sqrt{5^3} = \log_2(5^{\frac{3}{2}}) = \frac{3}{2} \cdot \log_2 5.$$

$\sqrt[m]{x^n} = x^{\frac{n}{m}}$

$\frac{1}{x^n} = x^{-n}$

Note

This property gives us the following special cases:

4a. $\log_a \frac{1}{x^n} = -n \cdot \log_a x$

4b. $\log_a \sqrt[m]{x^n} = \frac{n}{m} \cdot \log_a x.$

EXAMPLE 9 Write each sum as a single logarithm.

a. $(2 \cdot \log_3 a) + (3 \cdot \log_3 b) - \log_3 c$

b. $(\frac{1}{2} \cdot \log_2 a) + (3 \cdot \log_2 b) - (\frac{3}{2} \cdot \log_2 c)$

Solution We apply the property $\log_a(x^m) = m \cdot \log_a x$.

a. $(2 \cdot \log_3 a) + (3 \cdot \log_3 b) - \log_3 c = \log_3(a^2) + \log_3(b^3) + (-1) \cdot \log_3 c \Leftrightarrow$

$\log_3(a^2) + \log_3(b^3) + \log_3(c^{-1}) = \log_3(a^2 \cdot b^3 \cdot c^{-1}) = \log_3(\frac{a^2 \cdot b^3}{c})$

b. $(\frac{1}{2} \cdot \log_2 a) + (3 \cdot \log_2 b) - (\frac{3}{2} \cdot \log_2 c) = \log_2 a^{\frac{1}{2}} + \log_2 b^3 + (-\frac{3}{2}) \cdot \log_2 c \Leftrightarrow$

$\log_2 \sqrt{a} + \log_2 b^3 + \log_2 c^{(-\frac{3}{2})} = \log_2 \sqrt{a} + \log_2 b^3 + \log_2(\frac{1}{\sqrt{c^3}}) \Leftrightarrow$

$\log_2(\sqrt{a} \cdot b^3 \cdot \frac{1}{\sqrt{c^3}}) = \log_2(\frac{\sqrt{a} \cdot b^3}{\sqrt{c^3}})$

EXAMPLE 10 Calculate $\log_2 \sqrt[4]{2\cdot\sqrt{8\cdot\sqrt[3]{16}}}$.

Solution

$$\log_2 \sqrt[4]{2\cdot\sqrt{8\cdot\sqrt[3]{16}}} = \log_2 \sqrt[4]{2\cdot\sqrt{8\cdot 16^{\frac{1}{3}}}} = \log_2 \sqrt[4]{2\cdot\sqrt{2^3\cdot(2^4)^{\frac{1}{3}}}}$$

$$= \log_2 \sqrt[4]{2\cdot\sqrt{2^{3+\frac{4}{3}}}} = \log_2 \sqrt[4]{2\cdot\sqrt{2^{\frac{13}{3}}}} = \log_2 \sqrt[4]{2\cdot(2^{\frac{13}{3}})^{\frac{1}{2}}} = \log_2 \sqrt[4]{2^{1+\frac{13}{6}}}$$

$$= \log_2 \sqrt[4]{2^{\frac{19}{6}}} = \log_2(2^{\frac{19}{24}}) = \frac{19}{24}\cdot\underbrace{\log_2 2}_{1} = \frac{19}{24}$$

Property 5

The logarithm of the quotient of two positive numbers is equal to the difference between the logarithms of the dividend and the divisor to the same base:

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

Be careful!

$$\frac{\log_a x}{\log_a y} \neq \log_a x - \log_a y$$

Proof $\frac{x}{y} = a^{\log_a\left(\frac{x}{y}\right)}$. If we substitute $x = a^{\log_a x}$ and $y = a^{\log_a y}$ on the left side, we obtain

$$\frac{a^m}{a^n} = a^{m-n}$$

$$\frac{a^{\log_a x}}{a^{\log_a y}} = a^{\log_a\left(\frac{x}{y}\right)} \Leftrightarrow a^{\log_a x - \log_a y} = a^{\log_a\left(\frac{x}{y}\right)} \Leftrightarrow \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

For example,

$$\log_2 \frac{5}{3} = \log_2 5 - \log_2 3$$

$$\log_5(0.12) = \log_5\left(\frac{12}{100}\right) = \log_5\left(\frac{3}{25}\right) = \log_5 3 - \log_5(5^2) = \log_5 3 - 2$$

$$\log_2 10 + \log_2 4 - \log_2 5 = \log_2(10\cdot 4) - \log_2 5 = \log_2 40 - \log_2 5 = \log_2\frac{40}{5} = \log_2 8 = 3.$$

Notice that we can combine properties 4 and 5 to write expressions with addition and subtraction of logarithms as the logarithm of a single fraction. The addends form the numerator of the fraction and the subtrahends form the denominator, for example:

$$\log_a b + \log_a c - \log_a d + \log_a e - \log_a f = \log_a\left(\frac{b\cdot c\cdot e}{d\cdot f}\right).$$

As a numerical example, consider

$$\log_3 15 - \log_3 5 + \log_3 6 - \log_3 2 = \log_3\left(\frac{15\cdot 6}{5\cdot 2}\right) = \log_3 9 = \log_3(3^2) = 2.$$

Remember that this property only applies to logarithms with a common base.

EXAMPLE 11 Express log 30 and log $3.\bar{3}$ in terms of p given $\log 3 = p$.

Solution Since $30 = 3 \cdot 10$, we get $\log 30 = \log(3 \cdot 10) = \underbrace{\log 3}_{p} + \underbrace{\log 10}_{1} = p+1$.

Since $3.\bar{3} = \frac{10}{3}$, we have $\log 3.\bar{3} = \log \frac{10}{3} = \log 10 - \log 3 = 1 - p$.

EXAMPLE 12 Given $\log 300 = 2.47712$, calculate $\log(0.0027)$.

Solution $\log(0.0027) = \log(\frac{27}{10^4}) = \log 27 - \log 10^4 = \log 3^3 - 4 \cdot \log 10$

$= (3 \cdot \log 3) - 4$ (1)

$\log 10 = \log_{10} 10 = 1$

$\log 300 = \log(3 \cdot 100) = \log 3 + \log 10^2 = \log 3 + 2 \cdot \log 10 = 2 + \log 3$. So $\log 3 = \log 300 - 2$.

Using $\log 300 = 2.47712$, we get $\log 3 = 2.47712 - 2 = 0.47712$. (2)

Combining (1) and (2) gives us $\log(0.0027) = (3 \cdot 0.47712) - 4 = -2.56864$.

EXAMPLE 13 Write each logarithm as a sum or difference of logarithms to base a.

a. $\log_a \frac{b^3c^2}{d^4e^5}$ b. $\log_a \frac{\sqrt[5]{(b+c)^2}}{(d-e)^3}$

Solution a. $\log_a(\frac{b^3 \cdot c^2}{d^4 \cdot e^5}) = \log_a b^3 + \log_a c^2 - \log_a d^4 - \log_a e^5 = 3\log_a b + 2\log_a c - 4\log_a d - 5\log_a e$

b. We have $\log_a \frac{\sqrt[5]{(b+c)^2}}{(d-e)^3} = \log_a \sqrt[5]{(b+c)^2} - \log_a (d-e)^3 = \log_a (b+c)^{\frac{2}{5}} - 3\log_a (d-e)$

$= \frac{2}{5}\log_a (b+c) - 3\log_a (d-e)$.

Notice that logarithms cannot be distributed over addition or subtraction, and also that logarithms enable us to perform simpler operations (addition and subtraction) instead of multiplication and division. This is why logarithms are so useful in computation.

1. Changing the Base of a Logarithm

Most of the properties that we have seen so far only apply to logarithms to the same base. Now we will consider some key properties which allow us to deal with logarithms to different bases.

Property 6

Raising the base of a logarithm to a non-zero power is the same as dividing the logarithm by that power:

$$\log_{a^n} x = \frac{1}{n}\log_a x.$$

Proof $x = a^{\log_a x}$ and $x = (a^n)^{\log_{a^n} x} = a^{n\log_{a^n} x}$, so $a^{\log_a x} = a^{n\log_{a^n} x}$.

Since the bases are the same, we can equalize the exponents: $\log_a x = n\log_{a^n} x$.

Hence $\log_{a^n} x = \dfrac{1}{n}\log_a x$, as required.

For example, $\log_4 3 = \log_{2^2} 3 = \dfrac{1}{2}\log_2 3$ and $\log_{\frac{1}{3}} 9 = \log_{3^{-1}} 9 = \dfrac{1}{-1}\cdot\log_3 3^2 = -2$.

EXAMPLE 14 Write the following expression as a single logarithm to base 3.

$$\log_{\frac{1}{3}} 7 + 2\log_9 49 - \log_{\sqrt{3}} \frac{1}{7}$$

Solution Expressing the bases as powers of 3, we get

$$\log_{3^{-1}} 7 + 2\log_{3^2} 49 - \log_{3^{1/2}} \frac{1}{7} = \frac{1}{-1}\log_3 7 + 2\cdot\frac{1}{2}\cdot\log_3 49 - \frac{1}{\frac{1}{2}}\cdot\log_3 \frac{1}{7} \Leftrightarrow$$

$$-\log_3 7 + \log_3 7^2 - 2\log_3 7^{-1} = -\log_3 7 + 2\log_3 7 + 2\log 7 = 3\log_3 7 = \log_3 343.$$

Note

As a result of property 6,

6a. $\boxed{\log_{a^n} x^m = \dfrac{m}{n}\cdot\log_a x}$. 6b. $\boxed{\log_a x = \log_{a^n} x^n}$.

Check Yourself 2

Evaluate the expressions.

a. $\log_{36}\dfrac{1}{6} + \log_{\frac{1}{5}}\dfrac{1}{125} + \log_8 128 + \log_{\frac{1}{3}} 9$ b. $8^{\log_{\sqrt{2}} 3}$ c. $729^{\frac{1}{3}+\log_{81} 4}$

Answers

a. $\dfrac{17}{6}$ b. 729 c. 72

The following property gives us a another rule for changing the base of a logarithm.

Property 7 Let a, b and x be positive numbers such that $a \neq 1$ and $b \neq 1$. Then

$$\boxed{\log_a x = \frac{\log_b x}{\log_b a}}.$$

This formula is called the Change of Base formula.

Proof Let $\log_a x = y$. By definition, $a^y = x$.

So $x = b^{\log_b x}$ and $a = b^{\log_b a}$.

If we substitute these in $a^y = x$, we get

$$(b^{\log_b a})^y = b^{\log_b x} \Leftrightarrow b^{y \cdot \log_b a} = b^{\log_b x}.$$

Since the bases are the same on both sides, we can equalize the exponents:

$$y \cdot \log_b a = \log_b x \Leftrightarrow y = \frac{\log_b x}{\log_b a}.$$

Since $\log_a x = y$, we conclude $\log_a x = \dfrac{\log_b x}{\log_b a}$, as required.

For example,

$$\log_2 7 = \frac{\log_3 7}{\log_3 2} = \frac{\log 7}{\log 2} = \frac{\ln 7}{\ln 2}, \quad \log_3 5 = \frac{\log_5 5}{\log_5 3} = \frac{1}{\log_5 3},$$
$$\log 5 = \frac{\log_2 5}{\log_2 10} \text{ and } \ln x = \frac{\log_3 x}{\log_3 e}.$$

Scientific calculators do not have a $\boxed{\log_x y}$ button but they do have a $\boxed{\log}$ button (for base 10 logarithms) and an $\boxed{\ln}$ button (for natural logarithms). To calculate logarithms to a different base on a calculator, we therefore use the Change of Base formula.

Example: Find the value of $\log_2 7$ using a scientific calculator.

$\boxed{7}$ $\boxed{\log}$	0.84509804
$\boxed{/}$ $\boxed{2}$ $\boxed{\log}$	0.30102999
$\boxed{=}$	2.80735492

Conclusion

We can easily derive the following properties from the examples we have studied:

7a. $\boxed{\log_a b = \dfrac{1}{\log_b a}}$ and $\boxed{\log_a b \cdot \log_b a = 1}$ for $a, b > 0$ and $a, b \neq 1$.

7b. $\boxed{\dfrac{\log_a x}{\log_a y} = \dfrac{\log_b x}{\log_b y}}$ for $a, b, x, y > 0$ and $a, b \neq 1$.

EXAMPLE 15 Evaluate $\dfrac{4}{\log_2 12} + \dfrac{2}{\log_3 12}$.

Solution
$$\frac{4}{\log_2 12} + \frac{2}{\log_3 12} = 4 \cdot \frac{1}{\log_2 12} + 2 \cdot \frac{1}{\log_3 12} = 4\log_{12} 2 + 2\log_{12} 3$$
$$= \log_{12} 2^4 + \log_{12} 3^2 = \log_{12}(2^4 \cdot 3^2) = \log_{12} 12^2 = 2.$$

EXAMPLE 16 Calculate $\log_{20} 200$ in terms of p if $\log_5 2 = p$.

Solution We begin by changing $\log_{20}200$ to base 5:

$$\log_{20} 200 = \frac{\log_5 200}{\log_5 20} = \frac{\log_5(8 \cdot 25)}{\log_5(4 \cdot 5)} = \frac{\log_5 8 + \log_5 25}{\log_5 4 + \log_5 5} = \frac{\log_5 2^3 + \log_5 5^2}{\log_5 2^2 + 1} = \frac{3 \cdot \log_5 2 + 2}{2 \cdot \log_5 2 + 1}.$$

Using the substitution $\log_5 2 = p$, we obtain $\log_{20} 200 = \dfrac{3p+2}{2p+1}$.

EXAMPLE 17 $\log_a b = c$ is given. Express each logarithm in terms of c.

a. $\log_{a^2 b} ab^2$ b. $\log_{\frac{a^3}{b^2}} ab^4$

Solution In each case we use the Change of Base formula.

a. $\log_{a^2 b} ab^2 = \dfrac{\log_a ab^2}{\log_a a^2 b} = \dfrac{\log_a a + \log_a b^2}{\log_a a^2 + \log_a b} = \dfrac{1 + 2\log_a b}{2\log_a a + c} = \dfrac{1+2c}{2+c}$

b. $\log_{\frac{a^3}{b^2}} ab^4 = \dfrac{\log_a ab^4}{\log_a \dfrac{a^3}{b^2}} = \dfrac{\log_a a + \log_a b^4}{\log_a a^3 - \log_a b^2} = \dfrac{1 + 4\log_a b}{3\log_a a - 2\log_a b} = \dfrac{1+4c}{3-2c}$

EXAMPLE 18 Calculate each logarithm in terms of a, using the relation given.

a. $\log 25$; $a = \log 2$ b. $\log_3 18$; $a = \log_3 12$

c. $\log_{12} 27$; $a = \log_6 16$

Solution a. $\log 25 = \log 5^2 = 2\log 5 = 2\log \dfrac{10}{2} = 2(\log 10 - \log 2) = 2(1 - a) = 2 - 2a$

b. $\log_3 18 = \log_3(3^2 \cdot 2) = \log_3 3^2 + \log_3 2 = 2 + \log_3 2$. So we need to find $\log_3 2$ by using $\log_3 12 = a$:

$$\log_3 12 = \log_3(2^2 \cdot 3) = \log_3 2^2 + \log_3 3 = 2\log_3 2 + 1.$$

So $2\log_3 2 + 1 = a$, i.e. $\log_3 2 = \dfrac{a-1}{2}$.

In conclusion we can write $\log_3 18 = 2 + \log_3 2 = 2 + \dfrac{a-1}{2} = \dfrac{a+3}{2}$.

c. $\log_{12} 27 = \dfrac{\log_2 27}{\log_2 12} = \dfrac{\log_2 3^3}{\log_2 (2^2 \cdot 3)} = \dfrac{3\log_2 3}{\log_2 2^2 + \log_2 3} = \dfrac{3\log_2 3}{2 + \log_2 3}$ (1)

$\log_6 16 = \dfrac{\log_2 16}{\log_2 6} = \dfrac{\log_2 2^4}{\log_2 (3 \cdot 2)} = \dfrac{4}{\log_2 3 + \log_2 2} = \dfrac{4}{\log_2 3 + 1}$

Using the equality $\log_6 16 = \dfrac{4}{\log_2 3 + 1} = a$, we get $\log_2 3 = \dfrac{4-a}{a}$. (2)

Substituting (2) into (1) gives us $\log_{12} 27 = \dfrac{3(\frac{4-a}{a})}{2 + (\frac{4-a}{a})} = \dfrac{12 - 3a}{a + 4}$.

EXAMPLE 19 Simplify $\sqrt{25^{\frac{1}{\log_6 5}} + 49^{\frac{1}{\log_8 7}}}$.

Solution

$a^{\log_a x} = x$

$$\sqrt{25^{\frac{1}{\log_6 5}} + 49^{\frac{1}{\log_8 7}}} = \sqrt{25^{\log_5 6} + 49^{\log_7 8}} = \sqrt{5^{2\log_5 6} + 7^{2\log_7 8}} = \sqrt{5^{\log_5 6^2} + 7^{\log_7 8^2}} = \sqrt{6^2 + 8^2} = 10$$

EXAMPLE 20 Show that $\log_2 3 \cdot \log_5 7 \cdot \log_{11} 13 = \log_2 13 \cdot \log_5 3 \cdot \log_{11} 7$.

Solution If we use the Change of Base formula on the left-hand side with any base $a \neq 1$, we get

$$\log_2 3 \cdot \log_5 7 \cdot \log_{11} 13 = \frac{\log_a 3}{\log_a 2} \cdot \frac{\log_a 7}{\log_a 5} \cdot \frac{\log_a 13}{\log_a 11} = \frac{\log_a 13}{\log_a 2} \cdot \frac{\log_a 3}{\log_a 5} \cdot \frac{\log_a 7}{\log_a 11}$$

$$= \log_2 13 \cdot \log_5 3 \cdot \log_{11} 7.$$

Check Yourself 3

1. Evaluate the expressions.

   a. $\log_3 2 \cdot \log_4 3 \cdot \log_5 4 \cdot \log_6 5 \cdot \log_7 6 \cdot \log_8 7$

   b. $\dfrac{\log_3 135}{\log_{75} 3} - \dfrac{\log_3 45}{\log_{675} 3}$

2. Calculate each logarithm in terms of the variable(s) provided, using the given relation(s).

   a. $\log 40$; $\log 2 = a$

   b. $\log 45$; $\log 2 = a$ and $\log 3 = b$

   c. $\log_3 5$; $\log_6 2 = a$ and $\log_6 5 = b$

   d. $\log_{100} 40$; $\log_2 5 = a$

   e. $\log_{275} 60$; $\log_{12} 5 = a$ and $\log_{12} 11 = b$

   f. $\log_6 16$; $\log_{12} 27 = a$

3. Prove each statement by using the properties of logarithms.

a. $\log_{bn}(an) = \dfrac{\log_b a + \log_b n}{1 + \log_b n}$

b. $\dfrac{\log_a n}{\log_{ab} n} = 1 + \log_a b$

c. $\log_{ab} n = \dfrac{\log_a n \cdot \log_b n}{\log_a n + \log_b n}$

d. $\dfrac{1}{\log_2 N} + \dfrac{1}{\log_3 N} + \ldots + \dfrac{1}{\log_{2006} N} = 1$ if $N = 2006!$

$2006! = 2006 \cdot 2005 \cdot 2004 \cdot \ldots \cdot 2 \cdot 1$

Answers

1. a. $\dfrac{1}{3}$ b. -3

2. a. $2a + 1$ b. $1 - a + 2b$ c. $\dfrac{b}{1-a}$ d. $\dfrac{3+a}{2(a+1)}$ e. $\dfrac{1+a}{2a+b}$ f. $\dfrac{4(3-a)}{3+a}$

Property 8

$\boxed{b^{\log_a c} = c^{\log_a b}}$ for $a, b, c > 0$ and $a \neq 1$.

Proof $b^{\log_a c} = (a^{\log_a b})^{\log_a c} = (a^{\log_a c})^{\log_a b} = c^{\log_a b}$ because $a^{\log_a c} = c$.

So we obtain $b^{\log_a c} = c^{\log_a b}$.

For example, $7^{\log_2 5} = 5^{\log_2 7}$ and $3^{\log 2} = 2^{\log 3}$.

$(a^m)^n = (a^n)^m$

EXAMPLE 21 Calculate $2^{\log_3 5} - 5^{\log_3 2}$.

Solution By property 8, since $2^{\log_3 5} = 5^{\log_3 2}$, we have $2^{\log_3 5} - 5^{\log_3 2} = 5^{\log_3 2} - 5^{\log_3 2} = 0$.

EXAMPLE 22 Calculate $(2^{\log 27})^{\log_3 5} \cdot 8^{\log 2}$.

Solution Using $2^{\log 27} = 2^{\log 3^3} = (2^3)^{\log 3} = 8^{\log 3} = 3^{\log 8}$ and $\log_3 5 = \log_3 \dfrac{10}{2} = \log_3 10 - \log_3 2$,

we can write $(2^{\log 27})^{\log_3 5} \cdot 8^{\log 2} = (3^{\log 8})^{\log_3 10 - \log_3 2} \cdot 8^{\log 2}$

$$= \frac{(3^{\log 8})^{\log_3 10}}{(3^{\log 8})^{\log_3 2}} \cdot 8^{\log 2}$$

$$= \frac{(3^{\log_3 10})^{\log 8}}{(3^{\log_3 2})^{\log 8}} \cdot 8^{\log 2} = \frac{10^{\log 8}}{\cancel{2^{\log 8}}} \cdot \cancel{8^{\log 2}} = 8.$$

$(a^m)^n = (a^n)^m$
$3^{\log_3 10} = 10$
$3^{\log_3 2} = 2$
$10^{\log 8} = 8$
$8^{\log 2} = 2^{\log 8}$

EXERCISES 2.1

A. Basic Concept

1. Calculate the logarithms.

a. $\log_{1/3}\dfrac{1}{3}$ b. $\log_2\dfrac{1}{8}$ c. $\ln e$

d. $\log_{1/8} 1$ e. $\log_{\sqrt{2}} 2$ f. $\log\sqrt[5]{1000}$

g. $\log 1$ h. $\log(\ln e)$ i. $\ln\sqrt[3]{e}$

j. $\log_3(9\ln e^3)$ k. $\ln(\log 10^e)$ l. $\ln(\log 10)$

2. Solve each equation for x.

a. $3^x = 4$ b. $2^{x+1} = 3$ c. $3^{1-\frac{x}{2}} = 2$

d. $\sqrt{e^x} = 4$ e. $10^x = 5$ f. $10^{x-1} = 2$

3. Simplify the expressions.

a. $e^{\ln x}$ b. $10^{\log 3}$ c. $4^{\log_2 7}$

d. $5^{-\log_{25} 2}$ e. $27^{\log_{1/3} 4}$ f. $(x^{\log_3 5})^{\log_x 3}$

g. $x^{\log_x 3} + y^{\log_{1/y} 1/3} + z^{\log_z 1/3} - t^{\log_{1/t} 1/3}$

h. $-\log_2(\log_3\sqrt{\sqrt[4]{3}})$ i. $\left(\dfrac{16}{25}\right)^{\log_{\frac{125}{64}} 3}$

B. Types of Logarithm

4. Calculate the logarithms, using $\log 2 = 0.30103$ and $\log 3 = 0.4771$.

a. $\log 18$ b. $\log 30$ c. $\log\dfrac{1}{5}$

5. Find the number of digits in each number if $\log 2 = 0.30103$ and $\log 3 = 0.4771$.

a. 2^{50} b. 9^{10} c. 27^9 d. 18^{20}

C. Properties of Logarithms

6. Write each expression as a single logarithm.

a. $\dfrac{1}{3}\log x - \log y + \log z^2$

b. $-\dfrac{1}{2}\log x + \dfrac{1}{2}\log y + \dfrac{1}{2}\log z$

7. Write each expression as the sum or difference of the logarithms of a, b and c.

a. $\log(a^3b^2c)$ b. $\log(\sqrt[3]{a}\sqrt{bc})$

8. Evaluate the expressions.

a. $\log_{24} 4 + \log_{24} 6$

b. $\log 8 + \log 25 + \log 5$

c. $\log_{1/2}\dfrac{1}{4} + \log_5 625 + \log_{1/3} 81 + \log_{49}\dfrac{1}{7}$

d. $\log_2 1000 - \log_2 125$

9. Calculate each logarithm in terms of the variable(s) provided, using the given relation(s).

a. $\log_2 3$; $\log_3 2 = a$ b. $\log 25$; $\log 2 = a$

c. $\log_7 21$; $\log_3 7 = p$ d. $\log_3 18$; $\log_3 12 = a$

✪✪ e. $\log_{12} 60$; $\log_6 30 = a$ and $\log_{15} 24 = b$

10. $\log_x y = a$ is given. Express each logarithm in terms of a.

a. $\log_{x^3y^2} x^2y^3$ b. $\log_{x/y^2} x^3y^4$

11. Simplify the expressions.

a. $(\log_3 625\cdot\log_{1/5} 9) + (\log_4\dfrac{1}{125}\cdot\log_{1/25} 1024)$

b. $\log_{a^2} b^3\cdot\log_{b^3} c^4\cdot\log_{c^4} d^5\cdot\log_{d^5} a$

12. Find x in each case.

a. $\log_2 x = 3 - (2\cdot\log_2 3) + (3\cdot\log_2 5)$

b. $\log_3 x = 2 + (3\cdot\log_3 5) - (2\cdot\log_3 4)$

13. ✪ Prove each equality.

a. $\log_{x_1} x_2\cdot\log_{x_2} x_3\cdot\ldots\cdot\log_{x_n} x_1 = 1$

b. $x^{\log\frac{y}{z}}\cdot y^{\log\frac{z}{x}}\cdot z^{\log\frac{x}{y}} = 1$

14. ✪✪ Show that if $a^2 + b^2 = 7ab$ $(a, b > 0)$ then $\log\dfrac{a+b}{3} = \dfrac{1}{2}(\log a + \log b)$.

D. USING A TABLE OF LOGARITHMS

Before the invention of computers and calculators, mathematicians used values of logarithms printed in a logarithmic table to help them with their calculations. Today we can easily find the logarithm of any number to any base by using a calculator or computer. In this section we will learn how to use a logarithmic table for the same calculations.

Since most calculations involve decimal numbers, it makes sense to use a table of logarithms based on powers of 10. For this reason, logarithmic tables generally show logarithms to base 10. These logarithms are also known as common or Briggsian logarithms.

Henry Briggs published the first installment of his own table of common logarithms in 1617. The table contained the logarithms of all integers below 1000 to eight decimal places. He then published his *Arithmetica Logarithmica* in 1624, which contained the logarithms of all integers from 1 to 20,000 and from 90,000 to 100,000 to fourteen decimal places, together with an introduction which described the theory and use of logarithms. The interval from 20,000 to 90,000 was filled by Adriaan Vlacq, a Dutch mathematician. However, in Vlacq's table, which appeared in 1628, the logarithms were given to only ten decimal places. The only important published extension of Vlacq's table was made by Edward Sang in 1871, whose table contained the seven-place logarithms of all numbers below 200,000.

Edward Sang was a Scottish mathematician and engineer. His most remarkable achievement was his massive unpublished compilation of 26– and 15– place logarithmic, trigonometric and astronomic tables.

1. Parts of a Logarithm

The logarithm of a number which is not an exact power of 10 consists of two parts: a whole number, called the characteristic, and the decimal part, called the mantissa.

Definition

characteristic and mantissa of a logarithm

The common logarithm of any number x can be written as $\log x = c + m$ where $c \in \mathbb{Z}$ and $m \in [0, 1)$. In this notation, the integer c is called the characteristic and the positive real number m between 0 and 1 is called the mantissa of $\log x$.

For example, $\log 500 = 2.699 = 2 + 0.699$, so the characteristic of $\log 500$ is 2 and the mantissa is 0.699. We can find the characteristic of a logarithm by inspection, while the mantissa is obtained from a logarithmic table.

To find the characteristic of a logarithm, we use the following rules:

- The characteristic of the logarithm of a number x greater than 10 is positive, and is one less than the number of digits in x to the left of the decimal point.
- The characteristic of the logarithm of a number y less than 1 is negative, and is equal to the number of zeros to the left of the first non-zero digit in y.

Number	Scientific Notation	Characteristic
2	$2.0 \cdot 10^0$	0
41	$4.1 \cdot 10^1$	1
142.17	$1.4217 \cdot 10^2$	2
8450	$8.45 \cdot 10^3$	3
0.234	$2.34 \cdot 10^{-1}$	–1
0.0751	$7.51 \cdot 10^{-2}$	–2
0.00102	$1.02 \cdot 10^{-3}$	–3

Number	Scientific Notation	Mantissa
457	$4.57 \cdot 10^2$	0.6599
45.7	$4.57 \cdot 10^1$	0.6599
0.457	$4.57 \cdot 10^{-1}$	0.6599

A number is said to be expressed in scientific notation if it is in the form $a \cdot 10^b$ where $a \in \mathbb{R}$ and $1 \le a < 10$, and $b \in \mathbb{Z}$.

We can also find the characteristic of $\log x$ by writing x in scientific notation. The characteristic is the power of 10 when the number is written in scientific notation, as we can see in the table opposite.

We can find the mantissa of a logarithm by using a table of logarithms. Logarithms of numbers which have the same sequence of digits but a different decimal point have the same mantissa. For example, the logarithms of 457, 45.7, and 0.457 have the same mantissa because, when written in scientific form, the part next to the power of 10 in each is the same. The only differing part is the exponent of 10 which is the characteristic of the logarithm.

2. Using a Table of Logarithms

The table at the end of this section is a complete, four-place common logarithm table. 'Four-Place' means that the table shows the mantissas of common logarithms to four decimal places. To calculate a logarithm, we calculate its characteristic separately and then use the table to find the mantissa.

For example, imagine we want to calculate the common logarithm log 12.4. We begin by writing the number in scientific notation as $1.24 \cdot 10^1$. So the characteristic is 1. Then we look in the table to find the mantissa for 1.24: move down the left-hand column to find 1.2, and then across to the column labeled 4. The entry is 0934, and so the mantissa becomes 0.0934 (notice that the table only contains the decimal part of the mantissa, without the decimal point). The logarithm is therefore

$$\log 12.4 = \text{characteristic} + \text{mantissa} = 1 + 0.0934 = 1.0934.$$

As an exercise, use the table to verify that $\log 2.05 = 0.3118$, $\log 25 = 1.3979$, $\log 568 = 2.7543$ and $\log 0.00706 = -2.1512$.

Notice that log 0.00706 has a negative value. The minus sign applies only to the characteristic and not to the mantissa: $\log 0.00706 = -3 + 0.8488$ (not $-3 - 0.8488$). Sometimes the negative sign is written directly above the characteristic: $\log 0.00706 = \overline{3}.8488 = -3 + 0.8488$.

To find the logarithm of a number which is not in the table, we use a process called interpolation. For example, suppose we are asked to find the logarithm of 242.6. The mantissa for 2.426 is not listed in the table; however, the mantissas for 2.42 and 2.43 are. Since 2.426 lies between 2.420 and 2.430, its mantissa must lie between the mantissas for these two numbers; in fact, since 2.426 lies 6/10 of the way between 2.420 and 2.430, the mantissa must be 6/10 of the way between 0.3838 and 0.3856. Because the difference between the two numbers is 0.0018, and 6/10 of 0.0018 is 0.00108, we add 0.00108 to the mantissa of 2.420 to get 0.38488. The complete logarithm of 242.6, therefore, is 2.38488, rounded off to 2.3849.

We can also use a logarithmic table to find a number whose logarithm is given.

Definition

antilogarithm

The antilogarithm of a number a is the number whose logarithm is equal to a. In other words, the antilogarithm of $\log x$ is x. If $\log x = a$, we write $x = $ antilog a or $x = \log^{-1} a$ to mean that x is the antilogarithm of a.

To find an antilogarithm, we reverse the process for determining the logarithm. To find the antilogarithm of 3.9489, we identify the mantissa (0.9489) and the characteristic (3) then find the number corresponding to the mantissa in the table. It is in row 8.8 and column 9, and so it has the digits 8.89. We multiply this number by 10 to the power of the characteristic. So the antilogarithm is $8.89 \cdot 10^3$, which is 8890.

Notice that in some instances, we cannot find the exact mantissa in the table. We must then use the process of interpolation that we saw on the previous page.

EXAMPLE 23 Find the antilogarithm of each number.

a. 4.8768 b. $\overline{3}.9609$ c. 1.8771

Solution

a. The mantissa 0.8768 corresponds to the number 7.53 in the logarithmic table (row 7.5, column 3). Since the characteristic is 4, we write the antilogarithm as $7.53 \cdot 10^4$, which is 75300.

b. From the table of logarithms, we find the number 9.14 corresponds to the mantissa 0.9609. Since the characteristic is –3, the antilogarithm of $\overline{3}.9609$ is $9.14 \cdot 10^{-3} = 0.00914$.

c. We identify the characteristic 1 and the mantissa 0.8771, which is not listed in the logarithm table. So we use the interpolation method. Since 0.8771 is halfway between 0.8768 (= log 7.53) and 0.8774 (= log 7.54) in the table, the corresponding number for the given mantissa should be halfway between the numbers 7.53 and 7.54, i.e. 7.535. Thus we write the antilogarithm as $7.535 \cdot 10^1 = 75.35$.

Definition

cologarithm

The logarithm of the reciprocal of a positive number x is called the cologarithm of x, written as colog x. In other words, $\text{colog } x = \log 1/x = -\log x$.

EXAMPLE 24 Find the cologarithm of x in each case.

a. $\log x = 3.8182$ b. $\log x = 0.7404$ c. $\log x = \overline{5}.1959$

Solution To obtain a mantissa between zero and 1, we add and subtract 1 from the negative logarithms.

a. $\text{colog } x = -\log x = -3.8182 = -3 - 0.8182 = -3 - 1 + 1 - 0.8182 = -4 + 0.1818 = \overline{4}.1818$

b. $\text{colog } x = -\log x = -0.7404 = -1 + 1 - 0.7404 = -1 + 0.2596 = \overline{1}.2596$

c. $\text{colog } x = -\log x = -(-5 + 0.1959) = 5 - 0.1959 = 4.8041$

TABLE OF LOGARITHMS

	0	1	2	3	4	5	6	7	8	9
1.0	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
1.1	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
1.2	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
1.3	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
1.4	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
1.5	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
1.6	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
1.7	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
1.8	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
1.9	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
2.0	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
2.1	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
2.2	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
2.3	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
2.4	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
2.5	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
2.6	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
2.7	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
2.8	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
2.9	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
3.0	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
3.1	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
3.2	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
3.3	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
3.4	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
3.5	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
3.6	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
3.7	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
3.8	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
3.9	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
4.0	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
4.1	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
4.2	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
4.3	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
4.4	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
4.5	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
4.6	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
4.7	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
4.8	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
4.9	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
5.0	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
5.1	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
5.2	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
5.3	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
5.4	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
5.5	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
5.6	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
5.7	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
5.8	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
5.9	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
6.0	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
6.1	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
6.2	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
6.3	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
6.4	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
6.5	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
6.6	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
6.7	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
6.8	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
6.9	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
7.0	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
7.1	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
7.2	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
7.3	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
7.4	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
7.5	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
7.6	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
7.7	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
7.8	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
7.9	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
8.0	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
8.1	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
8.2	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
8.3	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
8.4	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
8.5	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
8.6	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
8.7	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
8.8	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
8.9	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
9.0	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
9.1	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
9.2	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
9.3	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
9.4	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
9.5	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
9.6	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
9.7	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
9.8	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
9.9	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

2 LOGARITHMIC FUNCTIONS

Now that we are familiar with the concept of logarithm and the properties of logarithms, we are ready to study logarithmic functions.

A. BASIC CONCEPT

Definition

basic logarithmic function

For $a > 0$, $a \neq 1$, the function $f: (0, \infty) \to \mathbb{R}$ defined by $f(x) = \log_a x$ is called a basic logarithmic function with base a.

In this section, the term 'logarithmic function' means a basic logarithmic function.

Notice that in a logarithmic function, the base is non-negative and the variable appears in the argument, not in the base. Accordingly, $f(x) = \log_3 x$, $g(x) = \log_8(7x + 2)$ $h(x) = \log_a x^3$, $a \in \mathbb{R}^+ \setminus \{1\}$ are all logarithmic functions, but $m(x) = \log_{-2} x$ and $n(x) = \log_x 7$ are not logarithmic functions.

- When the base of a logarithmic function is 10, the function is called a common logarithmic function, and is denoted by $f(x) = \log x$.
- If the base of a logarithmic function is the irrational number e then the function is called a natural logarithmic function, and is denoted by $f(x) = \ln x$.

Existence conditions for a logarithm:

$x > 0$

$f(x) = \log_a x$

$a \neq 1,\ a > 0$

Considering the definition of logarithm and the domain of a logarithmic function, we can see that any well-defined logarithm must satisfy two conditions: first, both the argument and the base must be positive, and second, the base cannot be 1. We call these conditions the existence conditions for logarithms.

EXAMPLE 25 Find the largest possible domain for each function.

a. $f(x) = \log_{\frac{1}{2}}(x - 2)$ b. $f(x) = \log_{\pi}(x^2 + x - 2)$ c. $f(x) = \log_x(x^2 - 1)$

Solution a. Since the base satisfies the existence conditions for a logarithm, we only need to identify the values of x that make the argument positive. Therefore, we need $x - 2 > 0$ which gives $x > 2$. So the domain of $f(x) = \log_{\frac{1}{2}}(x - 2)$ is $(2, \infty)$.

b. Similarly, we establish the condition $x^2 + x - 2 > 0$ and factorize this as $(x + 2) \cdot (x - 1) > 0$.

By making a sign table, or using any other method, we can identify the values of x which satisfy this quadratic inequality:

x	$-\infty$		-2		1		∞
$(x + 2)(x - 1)$		$+$	o	$-$	o	$+$	

So the domain of $f(x) = \log_{\pi}(x^2 + x - 2)$ is $\mathbb{R} \setminus [-2, 1]$, or $(-\infty, -2) \cup (1, \infty)$.

c. The given function is not a logarithmic function (can you see why?). However, we are simply asked to find the values of x which make the function well-defined. To find the domain of $f(x) = \log_x(x^2 - 1)$, since both the base and the argument depend on x, we check the conditions for both of them. In other words, we need to solve the system

$$\begin{cases} x^2 - 1 > 0 \\ x > 0 \quad \text{and} \quad x \neq 1. \end{cases}$$

From the first inequality we get $x < -1$ or $x > 1$. Combining this with $x > 0$ and $x \neq 1$, we find that $x > 1$ is the solution to the system, and so the domain of $f(x)$ is $(1, \infty)$.

Check Yourself 4

Find the largest domain for each function.

a. $f(x) = \log_3(\dfrac{2-x}{x+3})$

b. $f(x) = \log(\sqrt{3-2x} - 1)$

c. $f(x) = \log_{x+3} 7$

Answers

a. $(-3, 2)$

b. $(-\infty, 1)$

c. $(-3, \infty) \setminus \{-2\}$

B. GRAPHS OF LOGARITHMIC FUNCTIONS

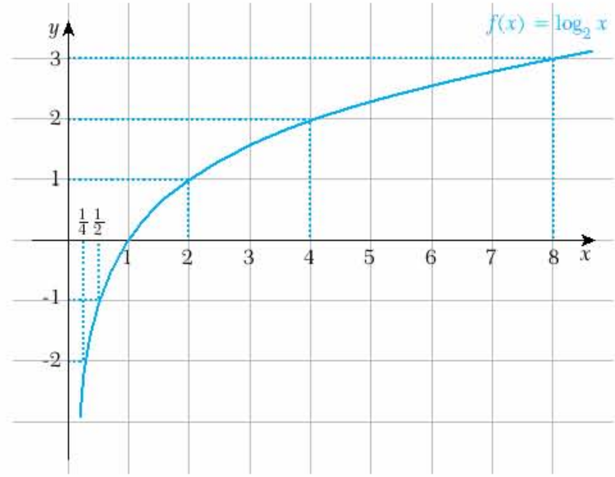
We can divide the graphs of logarithmic functions into two types, according to the value of the base.

1. Graph of $f(x) = \log_a x$ for $a > 1$

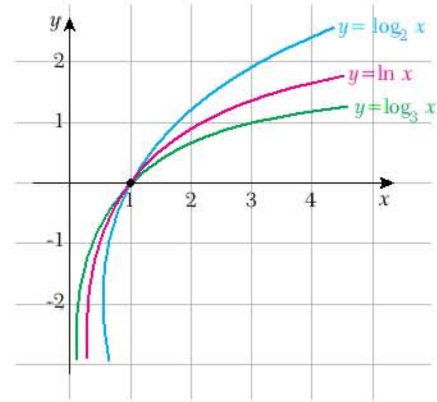
Consider the function $f(x) = \log_2 x$. If we rewrite the logarithmic function in its exponential form then we can graph it easily by plotting points. The exponential form of $y = \log_2 x$ is $2^y = x$, so we can make the following table:

$x = 2^y$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	$+\infty$
$y = \log_2 x$		-2	-1	0	1	2	3	

By drawing a smooth curve through the points (x, y) in the table, we obtain the graph opposite. As we can see the graph of the function approaches the negative y-axis but never touches it. So the y-axis is a vertical asymptote for the graph of $f(x)$.



The graph of any logarithmic function with base $a \in (1, \infty)$ is a curve similar to the curve shown above. The figure opposite shows the graphs $y = \log_2 x$, $y = \ln x$, and $y = \log_3 x$. We can see that as the base $a \in (1, \infty)$ increases, the curve moves closer to the positive x-axis and the negative y-axis.

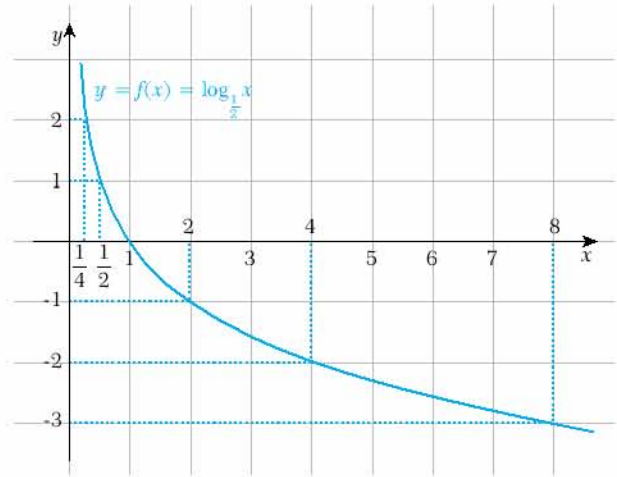


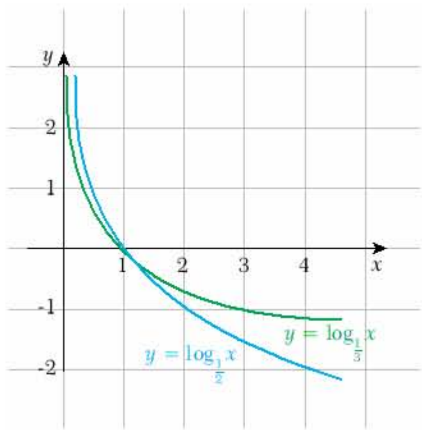
2. Graph of $f(x) = \log_a x$ for $a \in (0, 1)$

Now let us try to draw the graph of the logarithmic function $f(x) = \log_{\frac{1}{2}} x$. As before, we begin by writing the function in its exponential form: $(\frac{1}{2})^y = x$ where $y = \log_{\frac{1}{2}} x$. By picking convenient integer values of y and then calculating the corresponding x-values, we obtain the following table:

$x = (\frac{1}{2})^y$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	$+\infty$
$y = \log_{\frac{1}{2}} x$		2	1	0	-1	-2	-3	

The figure on the right shows a sketch of the graph $y = \log_{\frac{1}{2}} x$ obtained by plotting the points in the table. As before, we can see that the graph never touches or crosses the y-axis, and so the y-axis forms a vertical asymptote for the graph.



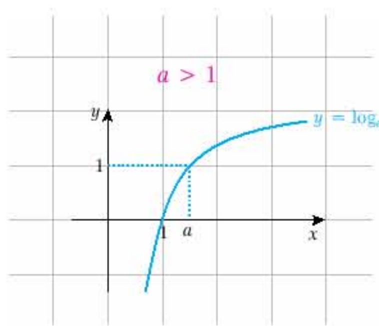


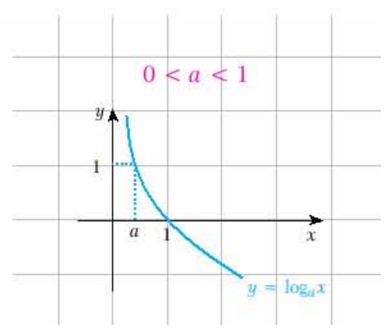
The graph of any logarithmic function with base $a \in (0, 1)$ is a curve similar to the curve shown above. The figure on the left shows the graphs $y = \log_{\frac{1}{2}} x$ and $y = \log_{\frac{1}{3}} x$.

We can conclude that as the base $a \in (0, 1)$ of a logarithmic function decreases, its graph gets closer to the positive x-axis and y-axis.

C. PROPERTIES OF LOGARITHMIC FUNCTIONS

Studying the graphs of basic logarithmic functions can help us to see some basic properties of these functions. Look at the graphs of two logarithmic functions $f: (0, \infty) \to \mathbb{R}$, $f(x) = \log_a x$:





Using the graphs we can identify the following properties of any logarithmic function of the form $f(x) = \log_a x$:

1. The domain of f is the set of positive real numbers.
2. The range of f is the set of real numbers.
3. The graph of f has x-intercept $(1, 0)$.
4. $(a, 1)$ is a point on this graph.
5. The y-axis $(x = 0)$ forms a vertical asymptote for the graph of f.

These two graphs also help us to understand the sign and inverse of a logarithmic function.

1. Sign of a Logarithmic Function

By looking at the graphs of the logarithmic functions in the previous section, we can deduce the following properties of the function $f(x) = \log_a x$:

$$\text{If } a > 1 \text{ then } \begin{cases} \log_a x < 0 \text{ for } 0 < x < 1 \\ \log_a x = 0 \text{ for } x = 1 \\ \log_a x > 0 \text{ for } x > 1. \end{cases}$$

$$\text{If } 0 < a < 1 \text{ then } \begin{cases} \log_a x > 0 \text{ for } 0 < x < 1 \\ \log_a x = 0 \text{ for } x = 1 \\ \log_a x < 0 \text{ for } x > 1. \end{cases}$$

Notice that in both cases, if both the base and the argument of a logarithm are from the same interval then the logarithm is positive, otherwise it is negative.

EXAMPLE 26 Identify whether the numbers are positive or negative.

a. $\log_{\frac{1}{2}} 0.9$ b. $\log 0.15$

Solution a. Since both $\frac{1}{2}$ and 0.9 are in $(0, 1)$, they are in the same interval and so $\log_{\frac{1}{2}} 0.9$ is positive.

b. The base (10) is in the interval $(1, \infty)$, but the argument (0.15) is in $(0, 1)$. Consequently, $\log 0.15 < 0$.

We can find the sign of a logarithmic function with a more complicated argument by substituting x for the argument and using the same rule. Let us look at some examples.

EXAMPLE 27 Identify the sign of each function with respect to x.

a. $f(x) = \log_3 x$ b. $f(x) = \log_{\frac{1}{2}}(x-1)$

Solution a. The base of the function is 3 which is greater than 1, so we use the rule for $a > 1$:

$$f(x) < 0 \text{ for } 0 < x < 1,\ f(x) = 0 \text{ for } x = 1, \text{ and } f(x) > 0 \text{ for } x > 1.$$

b. By the rule, $\log_{\frac{1}{2}}(x-1)$ is $\begin{cases} \text{positive for } 0 < x-1 < 1 \\ \text{zero for } x-1 = 1 \\ \text{negative for } x-1 > 1 \end{cases} \Leftrightarrow \begin{cases} f(x) > 0 \text{ for } 1 < x < 2 \\ f(x) = 0 \text{ for } x = 2 \\ f(x) < 0 \text{ for } x > 2. \end{cases}$

EXAMPLE 28 Find the greatest possible domain of each function.

a. $f(x) = \log_3(\log_2 x)$ b. $f(x) = \log_{\frac{1}{3}}(\log_9(2x+4))$

Solution a. The function f exists if $\begin{cases} x > 0 \text{ (i.e. } \log_2 x \text{ exists)} \\ \log_2 x > 0 \text{ (i.e. } \log_3(\log_2 x) \text{ exists)} \end{cases} \Leftrightarrow \begin{cases} x > 0 \\ x > 1. \end{cases}$

The solution for this system is $x \in (1, \infty)$, which is the greatest domain of $f(x)$.

b. The greatest domain is the solution set of the system

$$\begin{cases} 2x+4 > 0 \\ \log_9(2x+4) > 0 \end{cases} \Leftrightarrow \begin{cases} x > -2 \\ 2x+4 > 1 \end{cases} \Leftrightarrow \begin{cases} x > -2 \\ x > -\frac{3}{2} \end{cases} \Leftrightarrow x \in (-\frac{3}{2}, \infty).$$

Check Yourself 5

1. Identify the sign of each function with respect to x.

a. $f(x) = \log_{\frac{1}{2}}(3x-1)$ b. $f(x) = \log_{\frac{1}{3}} \dfrac{x-2}{3-x}$

2. Determine the greatest domain of each function.

a. $f(x) = \log_{\frac{1}{3}}(\log_{\frac{1}{5}} x)$ b. $f(x) = \log_{\frac{1}{2}}(\log \dfrac{x}{x+1})$

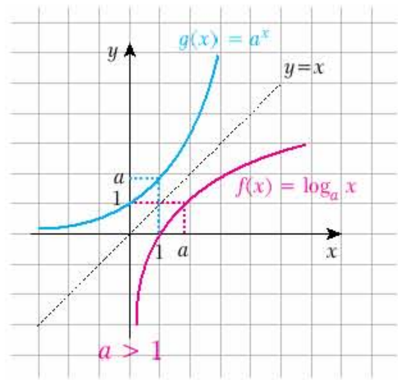
Answers

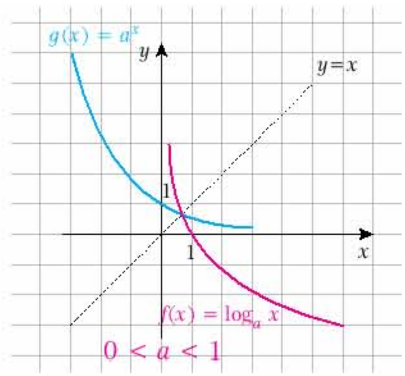
1. a. $f(x)$ is $\begin{cases} \text{positive} & \text{for} & x \in (\frac{1}{3}, \frac{2}{3}) \\ \text{zero} & \text{for} & x = \frac{2}{3} \\ \text{negative} & \text{for} & x \in (\frac{2}{3}, \infty). \end{cases}$ b. $f(x)$ is $\begin{cases} \text{positive} & \text{for} & x \in (2, \frac{5}{2}) \\ \text{zero} & \text{for} & x = \frac{5}{2} \\ \text{negative} & \text{for} & x \in (\frac{5}{2}, 3). \end{cases}$

2. a. $(0, 1)$ b. $(-\infty, -1)$

2. Inverse of a Logarithmic Function

Recall that we defined logarithms so that we could write the inverse of a^x in a convenient way. Thus the logarithmic function $f(x) = \log_a x$ and the exponential function $g(x) = a^x$ are inverses of each other. This is why their graphs are reflections of each other in the line $y = x$, as we can see below.





We can also prove this inverse property formally, as follows:

By the property of an inverse function, the functions f: $(0, \infty) \to \mathbb{R}$, $f(x) = \log_a x$ and g: $\mathbb{R} \to (0, \infty)$, $g(x) = a^x$ are inverse functions if their composition is the identity function $I(x)$. In other words,

$$(f \circ g)(x) = (g \circ f)(x) = I(x) = x \Leftrightarrow f(x) = g^{-1}(x) \text{ and } g(x) = f^{-1}(x).$$

Let us check that this is true.

$$(f \circ g)(x) = f(g(x)) = (\log_a x) \circ (a^x) = \log_a(a^x) = x \log_a a = x \text{ for all } x \in \mathbb{R}, \text{ and}$$

$$(g \circ f)(x) = g(f(x)) = (a^x) \circ (\log_a x) = a^{(\log_a x)} = x \text{ for all } x \in (0, \infty).$$

So $f(x) = \log_a x$ and $g(x) = a^x$ are indeed inverse functions.

Note that as a result, $f(x) = \log x$ and $g(x) = 10^x$ are inverse functions. $f(x) = \ln x$ and $g(x) = e^x$ are also inverse functions.

Since only bijective functions have an inverse, and logarithmic and exponential functions are inverses of each other, we can conclude that logarithmic functions are bijective. In other words, logarithmic functions are both one-to-one and onto.

We can use the one-to-one property of logarithmic functions to solve equations involving logarithms. For now it is enough to state that

$$\boxed{\log_a x = \log_a y \Leftrightarrow x = y}.$$

For example, if $\log_2 x = \log_2 5$ then $x = 5$. We will look at this property in more detail in Chapter 3.

EXAMPLE 29 Find the inverse of $f: \mathbb{R} \to (0, \infty)$, $f(x) = 2^x$.

Solution The inverse is $f^{-1}: (0, \infty) \to \mathbb{R}$, $f^{-1}(x) = \log_2 x$.

EXAMPLE 30 Find the inverse of $f: \mathbb{R} \to (0, \infty)$, $f(x) = 3^{x+1}$.

Solution $y = 3^{x+1} \Leftrightarrow x + 1 = \log_3 y \Leftrightarrow x = \log_3 y - 1$.

Interchanging x and y gives us the inverse of the given function:

$f^{-1}: (0, \infty) \to \mathbb{R}$, $f^{-1}(x) = \log_3 x - 1$.

EXAMPLE 31 Find the inverse of $f: (1, \infty) \to \mathbb{R}$, $f(x) = 1 - \log_2(x - 1)$.

Solution Isolate x on one side of the equation $y = f(x)$:

$$y = 1 - \log_2(x-1) \Leftrightarrow \log_2(x-1) = 1 - y \Leftrightarrow x - 1 = 2^{1-y} \Leftrightarrow x = 2^{1-y} + 1.$$

Interchange x and y: $y = 2^{1-x} + 1$.

Write the inverse function: $f^{-1}: \mathbb{R} \to (0, \infty)$, $f^{-1}(x) = 2^{1-x} + 1$.

EXAMPLE 32 Find the inverse of each function.

a. $f(x) = 1 - 3^{1-2x}$ b. $f(x) = \ln(x + 2)$

Solution a. Starting with $y = 1 - 3^{1-2x}$ and rewriting in terms of x, we get

$$y = 1 - 3^{1-2x} \Leftrightarrow 3^{1-2x} = 1 - y \Leftrightarrow 1 - 2x = \log_3(1-y) \Leftrightarrow$$

$$2x = \log_3 3 - \log_3(1-y) \Leftrightarrow 2x = \log_3\left(\frac{3}{1-y}\right) \Leftrightarrow x = \frac{1}{2}\cdot\log_3\left(\frac{3}{1-y}\right) \Leftrightarrow x = \log_3\sqrt{\frac{3}{1-y}}.$$

$y = a^x \Leftrightarrow x = \log_a y$

$\log_a x - \log_a y = \log_a \frac{x}{y}$

So $f^{-1}(x) = \log_3\sqrt{\frac{3}{1-x}}$ and $f^{-1}\ (-\infty, 1) \to \mathbb{R}$.

b. We can easily identify $f: (-2, \infty) \to \mathbb{R}$. To find the inverse $f^{-1}: \mathbb{R} \to (-2, \infty)$, we use $y = \ln(x + 2) \Leftrightarrow x + 2 = e^y \Leftrightarrow x = e^y - 2 \Leftrightarrow f^{-1}(x) = e^x - 2$.

Check Yourself 6

1. Find the inverse of each function $f: \mathbb{R} \to (0, \infty)$.

   a. $f(x) = 3^x$ b. $f(x) = (\frac{1}{5})^{x-1}$ c. $f(x) = e^{-x}$

2. Find the inverse of each function and determine its domain and range.

   a. $f(x) = 2^{3x-2}$ b. $f(x) = 1 + 3\log x$

Answers

1. a. $f^{-1}(x) = \log_3 x$ b. $f^{-1}(x) = \log_{\frac{1}{5}} \frac{x}{5}$ c. $f^{-1}(x) = \ln \frac{1}{x}$

2. a. $f^{-1}: (0, \infty) \to \mathbb{R},\ f^{-1}(x) = \log_2 \sqrt[3]{4x}$ b. $f^{-1}: \mathbb{R} \to (0, \infty),\ f^{-1}(x) = \sqrt[3]{10^{x-1}}$

3. Monotone Property of Logarithmic Functions

The graphs of logarithmic functions suggest the following properties:

If $a > 1$ then $f(x) = \log_a x$ is strictly increasing.

If $0 < a < 1$ then $f(x) = \log_a x$ is strictly decreasing.

We will look at logarithmic inequalities in more detail in the next chapter. For now, it is enough to remember two important rules:

1. Since $f(x) = \log_a x$ is an increasing function when $a > 1$, we can write

$$x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2),\ \text{i.e. } x_1 < x_2 \Leftrightarrow \log_a x_1 < \log_a x_2 \text{ for } a > 1.$$

   In other words, if we take the logarithm of both sides of an inequality to the same base $a > 1$, the direction of the inequality will stay the same.

   For example, $2 < 3$, so $\log_5 2 < \log_5 3$ because $a = 5 > 1$.

2. Since $f(x) = \log_a x$ is strictly decreasing when $0 < a < 1$, we have

$$x_1 < x_2 \Leftrightarrow f(x_1) > f(x_2),\ \text{i.e. } x_1 < x_2 \Leftrightarrow \log_a x_1 > \log_a x_1.$$

   In other words, if we take the logarithms of both sides of an inequality to the same base $a \in (0, 1)$, the direction of the inequality will be reversed.

   For example, $2 < 3$ so $\log_{\frac{1}{2}} 2 > \log_{\frac{1}{2}} 3$.

EXAMPLE 33 State the monotony (strictly increasing or strictly decreasing) of each function.

a. $f(x) = \log_{\frac{1}{3}} x$ b. $f(x) = \log x$

Solution a. Since the base $a = \dfrac{1}{3}$ is between 0 and 1, $f(x) = \log_{\frac{1}{3}} x$ is strictly decreasing.

b. $\log x$ is a common logarithm with base $a = 10$ which is greater than 1. So $f(x) = \log x$ is a strictly increasing function.

EXAMPLE 34 State the monotony of each function.

a. $f(x) = \log_{\frac{1}{2}}(x-1)$ b. $f(x) = \log_{\frac{1}{3}}(-2x-1)$

Solution a. $f(x) = \log_{\frac{1}{2}}(x-1) = \overbrace{(\log_{\frac{1}{2}} x)}^{g(x)} \circ \overbrace{(x-1)}^{h(x)}$ where $g(x) = \log_{\frac{1}{2}} x$ is strictly decreasing and $h(x) = x - 1$ is strictly increasing. Since $g(x)$ and $h(x)$ have different monotonies, $f(x)$ is strictly decreasing.

b. Similary, $f(x) = \log_{\frac{1}{3}}(-2x-1)$ can be written as the composition of $g(x) = \log_{\frac{1}{3}} x$ and $h(x) = -2x - 1$. Since both of these functions are strictly decreasing, $f(x)$ is a strictly decreasing function.

If two functions have different monotonies then their composition is strictly decreasing, If two functions are both strictly decreasing (or increasing) then their composition is decreasing (or increasing).

EXAMPLE 35 Compare the numbers in each pair.

a. $\log_2 5$, $\log_2 7$ b. $\log_6 4$, $\log_5 4$

Solution a. Since $2 > 1$ and $\log_2 x$ is strictly increasing, the direction of the inequality between the logarithms is the same as the direction of the inequality between the arguments: $5 < 7$ so $\log_2 5 < \log_2 7$.

b. Since the logarithms do not have the same base, we first apply the formula $\log_a b = \dfrac{1}{\log_b a}$ to obtain logarithms with the same base:

$\log_6 4 = \dfrac{1}{\log_4 6}$ and $\log_5 4 = \dfrac{1}{\log_4 5}$.

Since $\log_4 x$ is an increasing function, we get $\log_4 6 > \log_4 5 > 0$. Therefore we have

$$\log_4 6 > \log_4 5 \Leftrightarrow \frac{1}{\log_4 6} < \frac{1}{\log_4 5} \Leftrightarrow \log_6 4 < \log_5 4.$$

EXAMPLE 36 Compare a and b given each inequality.

a. $\log_2 a < \log_2 b$ b. $\log_a \frac{1}{2} < \log_b \frac{1}{2}$

Solution a. Since the function $y = \log_2 x$ is an increasing function, the arguments compare in the same way as the logarithms: $\log_2 a < \log_2 b \Leftrightarrow a < b$.

b. We can rewrite $\log_a \frac{1}{2} < \log_b \frac{1}{2}$ as $\frac{1}{\log_{\frac{1}{2}} a} < \frac{1}{\log_{\frac{1}{2}} b} \Leftrightarrow \log_{\frac{1}{2}} a > \log_{\frac{1}{2}} b$.

Since $\log_{\frac{1}{2}} x$ is a decreasing function, we conclude $\log_{\frac{1}{2}} a > \log_{\frac{1}{2}} b \Leftrightarrow a < b$.

EXAMPLE 37 In each case, find the interval between two consecutive integers in which the number lies.

a. $\log_2 15$ b. $\log_{\frac{1}{2}} \frac{1}{5}$ c. $\log 1720$

Solution a. In problems like this we begin by writing the argument of the logarithm between two consecutive powers of the base: $2^3 < 15 < 2^4$.

If we take the logarithms of all these numbers to base 2, the direction of the inequalities will remain the same because the base is greater than 1. In other words,

$\log_2 2^3 < \log_2 15 < \log_2 2^4$ and so $3 < \log_2 15 < 4$. So the interval is (3, 4).

b. Since $(\frac{1}{2})^3 < \frac{1}{5} < (\frac{1}{2})^2$ and the base is between zero and 1,

we get $\log_{\frac{1}{2}}(\frac{1}{2})^3 > \log_{\frac{1}{2}} \frac{1}{5} > \log_{\frac{1}{2}}(\frac{1}{2})^2 \Leftrightarrow 3 > \log_{\frac{1}{2}} \frac{1}{5} > 2$. So the interval is (2, 3).

c. Using the same approach as above, we obtain $\log 10^3 < \log 1720 < \log 10^4$. Therefore, $3 < \log 1720 < 4$.

Check Yourself 7

1. Determine whether each function is strictly increasing (↗) or strictly decreasing (↘).
   a. $f(x) = \ln(3x)$ b. $f(x) = \log_2 x + \log_3 x$
   c. $f(x) = \log_{\frac{1}{2}}(2x+3)$ d. $f(x) = \log_2(2^x + x)$
2. Find the integer part of each number.
   a. $\log_3 35$ b. $3\log_2 5$ c. $\log \pi$ d. $\log_{\frac{1}{2}} 5$ e. $\log_{\frac{1}{3}}(\frac{1}{25})$
3. Find the bigger number in each pair.
   a. $\log_2 5$, $\log_2 7$ b. $\log_{\frac{1}{3}} 2$, $\log_{\frac{1}{3}} 5$ c. $\log_3 4$, $\log_4 3$ d. $\log_2 3$, $\log_3 5$

4. Determine whether each number is positive or negative.

a. $\log 0.5$ b. $\ln 2$ c. $\log_{\frac{1}{3}} \frac{1}{2}$ d. $\log_{\frac{1}{2}} 3+1$ e. $\log_2 7 - 2$

5. Prove the inequalities.

a. $\log_3 4 < \frac{3}{2} < \log_2 3$ b. $\log_4 5 + \log_5 6 + \log_6 7 < 3\log_4 5$

Answers

1. a. ↗ b. ↗ c. ↘ d. ↗
2. a. 3 b. 6 c. 0 d. –2 e. 2
3. a. $\log_2 7$ b. $\log_{\frac{1}{3}} 2$ c. $\log_3 4$ d. $\log_2 3$
4. a. negative b. positive c. positive d. negative e. positive
5. a. (Hint: $\log_3 16 < \log_3 27$ and $\log_2 8 < \log_2 9$) b. (Hint: $\log_6 7 < \log_5 6 < \log_4 5$)

EXERCISES 2.2

A. Basic Concept

1. For which values of x is each logarithm defined?

a. $\log_2(x-2)$
b. $\log_{\frac{1}{3}}(x^2-5x-6)$
c. $\ln(x^2-2x+1)$
d. $\log_{\frac{1}{3}}|x|$
e. $\log_{1.5}\dfrac{1-x}{x+1}$
f. $\log_2\dfrac{x\cdot(x-2)}{3-x}$
g. $\log_{x^2-2x+2} e$
h. $\log_{x+3}(x^2-x)$

2. State the domain of each function.

a. $f(x)=\log_3(x-2)$
b. $f(x)=\log_2(x-1)$
c. $f(x)=\log_{\frac{1}{3}}(\log_{\frac{1}{2}}x)$
d. $f(x)=\log_x(2x-1)$
e. $f(x)=\ln\dfrac{1-x}{x+2}$
f. $f(x)=\log_2\sqrt{x^2-4}$
g. $f(x)=\log\sqrt{x^2-x}$
h. $f(x)=\log_{2-x}(3x-4)$
i. $f(x)=\log_9(\log_{\frac{1}{3}}(2x+4))$
j. $f(x)=\log_4(x^2+x-2)$
k. $f(x)=\log(x-2)+\log(3-x)$
l. $f(x)=\log_2(\log_2(1-x))$

B. Graphs of Logarithmic Functions

3. Sketch each graph by plotting selected points.

a. $y=\log_5 x$
b. $y=\log_{\frac{1}{3}} x$
c. $y=\log x$
d. $y=\ln x$

C. Properties of Logarithmic Functions

4. Identify the sign of each function with respect to x.

a. $f(x)=\log_3(2x-1)$
b. $f(x)=\ln|x-1|$

5. Write > or < in each gap to make true statements.

a. $\log_3 2 \ldots 1$
b. $\log_{0.13} 0.14 \ldots 0$
c. $\log_4 5 \ldots 1$
d. $\log_{1/2} 0.3 \ldots 2$

6. Order x, y and z in each case.

a. $x=\log_2 3,\quad y=\log_2 5,\quad z=\log_2 7$
b. $x=\log_{1/2} 3,\quad y=\log_{1/2} 5,\quad z=\log_{1/2} 7$
c. $x=\log_2 3,\quad y=\log_3 4,\quad z=\log_4 5$

7. Find the inverse of each function.

a. $f(x)=5^x$
b. $f(x)=(\frac{2}{3})^x$
c. $f(x)=e^x$
d. $f(x)=5^{-x}$
e. $f(x)=2^{x-1}$
f. $f(x)=1+3^{2x-1}$
g. $f(x)=2-e^{x+1}$
h. $f(x)=\log_2(x-3)$
i. $f(x)=\ln(x+2)$
j. $f(x)=3\log(x+1)$
k. $f(x)=\log 10x$
l. $f(x)=\log_3(x-4)$

8. Determine whether each function is strictly increasing or decreasing.

a. $f(x)=\log_2(1-3x)$
b. $f(x)=\log_{\frac{1}{5}}(2x+1)$

Mixed Problems

9. $f(x)=\sqrt{x}+5$, $g(x)=\log_3(5x+1)$ and $(f\circ g^{-1})(4)=m^2$ are given. What is m?

10. Compare a and b in each case.

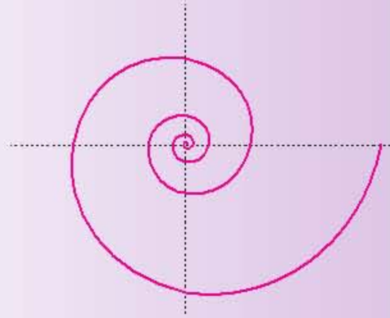
a. $a=\log_{\frac{1}{3}}\frac{1}{4}$, $b=\log_{\frac{1}{3}}\frac{1}{5}$
b. $a=\log_4 3$, $b=\log_5 3$

11. Given $f(x)=\log_2 x$, calculate $f(\frac{4}{x^2})+2\cdot f(x)$.

12. Find a if $f(x)=\log_3(4x+a)$ and $f^{-1}(4)=2$.

LOGARITHMIC SPIRALS

A logarithmic spiral is a special kind of spiral curve which often appears in nature. It is defined by the polar equation $r = a \cdot e^{b \cdot \theta}$ where r is the distance of the curve from the origin, θ is the angle of the curve to the x-axis, and a and b are arbitrary constants. The logarithmic relation between the angle of the spiral and its radius ($\theta = \frac{1}{b} \cdot \ln \frac{r}{a}$) gives the spiral its name. The logarithmic spiral is also known as the *equiangular spiral* or *growth spiral*.



Evangelista Torricelli

Torricelli was an Italian scientist who was the first man to create a sustained vacuum and to discover the principle of a barometer. He also achieved some important results in the development of calculus.

The logarithmic spiral was first described by Rene Descartes in 1638. The Italian scientist Torricelli worked on the curve independently, and found the curve's length. The spiral was later studied by Jakob Bernoulli (1654-1705), who called it *Spira Mirabilis*, 'the marvelous spiral'. Bernoulli was so fascinated by the spiral that he wanted to have one engraved on his headstone when he died, together with the Latin words *eadem mutata resurgo*, which mean 'although changed, still remaining the same'. Unfortunately, the engraver did not follow Bernoulli's wishes completely: he engraved an Archimendean speral instead.

Logarithmic spirals appear in many parts of nature where growth is proportional to the size of an organism. One example is the Nautilus shell, which is formed by a kind of mollusc.

The arms of spiral galaxies often have the shape of a logarithmic spiral. The arms of tropical cyclones such as hurricanes also show a roughly logarithmic spiral pattern.

Insects approach a light source in a logarithmic spiral because they are used to having the light source at a constant angle to their flight path. Similarly, hawks approach their prey in a logarithmic spiral: their sharpest view is at an angle to their direction of flight.

3 SIMPLE VARIATIONS OF LOGARITHMIC FUNCTIONS (OPTIONAL)

Now that we are familiar with basic logarithmic functions and their graphs, we are ready to study the properties and graphs of general logarithmic functions.

Definition

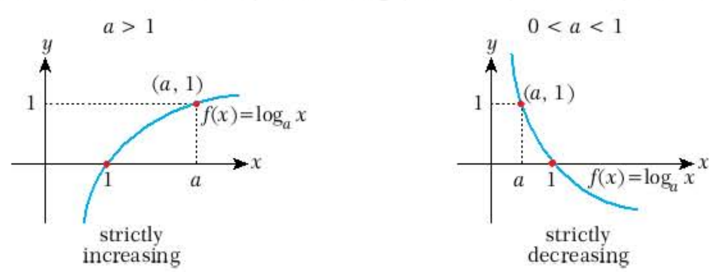
logarithmic function (general form)

A function of the form $f(x) = c \cdot \log_a|d(x + p)| + k$ where a, c, p, d and k are real numbers with $c, d \neq 0$ is called a logarithmic function with base a.

We can sketch the graph of a logarithmic function by plotting a selection of points, but this requires complicated calculations for finding the points. Alternatively, we can use some simple strategies to sketch the graph of a general logarithmic function with little or no computation.

Let us first recall the common properties of all basic logarithmic functions ($f(x) = \log_a x$):

1. The domain is $(0, \infty)$.
2. The range is $\mathbb{R}$.
3. The graph does not cross the y-axis. The y-axis is a vertical asymptote.
4. The graph has an x-intercept at $(1, 0)$.
5. $(a, 1)$ is also a point on the graph.
6. The function is either strictly increasing (for $a > 1$) or strictly decreasing (for $a \in (0, 1)$):



We can use all of these observations to roughly sketch the graph of any basic logarithmic function $f(x) = \log_a x$.

One method for drawing a rough graph of a general logarithmic function $g(x) = c \cdot \log_a|d(x + p)| + k$ is to start with the graph of $f(x) = \log_a x$ and apply transformations (horizontal or vertical shift, shrink, stretch or reflection) to $f(x)$ step by step. This is similar to the approach that we used to sketch the graph of exponential functions.

We can summarize the main transformations as follows:

For a function $g(x) = c \cdot \log_a|d(x + p)| + k$, the constants c, a, d, p and k have the following effect on the graph of $f(x) = \log_a x$:

- k represents a vertical shift k units up if $k > 0$, or $|k|$ units down if $k < 0$.
- p represents a horizontal shift p units to the left if $p > 0$, or $|p|$ units to the right if $p < 0$.
- c represents a vertical stretch, shrink or reflection:
  - $c < 0$ means a reflection in the x-axis.
  - $|c| > 1$ means a vertical stretch by a factor of $|c|$.
  - $0 < |c| < 1$ means a vertical shrink by a factor of $|c|$.
- d represents a horizontal stretch, shrink or reflection:
  - $d < 0$ means a reflection in the y-axis.
  - $0 < |d| < 1$ means a horizontal stretch by a factor of $\frac{1}{|d|}$.
  - $|d| > 1$ means a horizontal stretch by a factor of $\frac{1}{|d|}$.

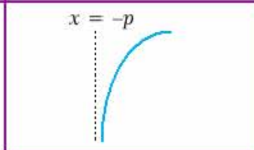
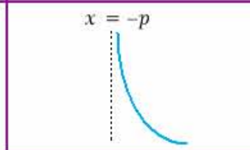
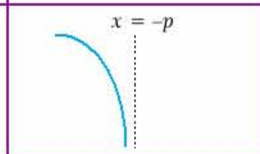
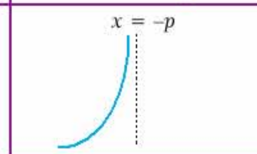
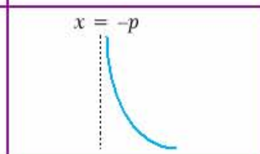
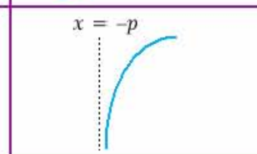
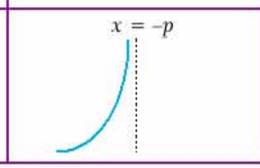
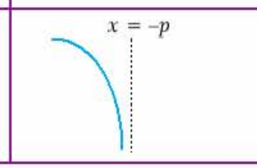
In addition, any point (x, y) on the graph of $f(x) = \log_a x$ will move to $(\frac{x}{d} - p,\ c \cdot y + k)$ on the graph of $g(x) = c \cdot \log_a |d(x + p)| + k$.

We mostly use these transformations to sketch the graphs of logarithmic functions containing absolute values.

We can alternatively sketch the graph of $g(x) = c \cdot \log_a|d(x + p)| + k$ by identifying the vertical asymptote $x = -p$ and two points which lie on the graph. To identify the points, we consider the fact that any point (x, y) on the graph $y = f(x)$ moves to $(\frac{x}{d} - p,\ c \cdot y + k)$ on the graph $y = c \cdot f|d(x + p)| + k$, and so

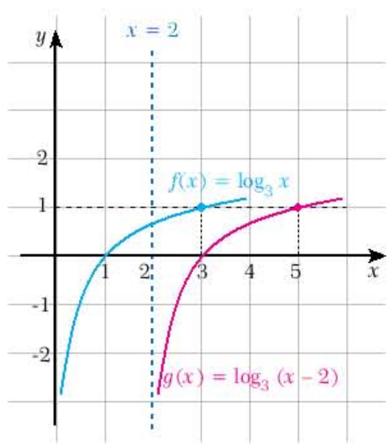
$y = \log_a x$		$y = c \cdot \log_a|d(x + p)| + k$
$(1, 0)$	$\longrightarrow$	$(\frac{1}{d} - p,\ k)$
$(a, 1)$	$\longrightarrow$	$(\frac{a}{d} - p,\ c + k)$.

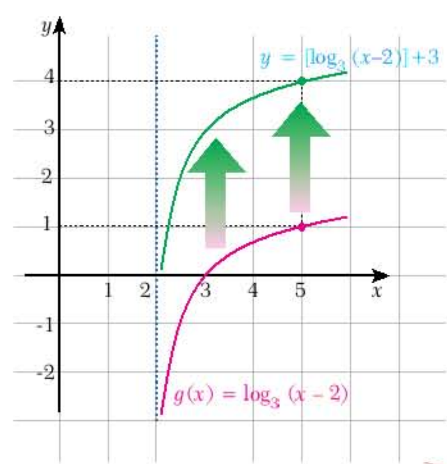
The following table is helpful when using this second approach to sketching a graph:

Graphs of functions of the form $f(x) = c \cdot \log_a\lvert d(x+p)\rvert + k$, where $d \neq 0$					
c	d	$a > 1$	$0 < a < 1$	Domain	Range
+	+			$(-p, \infty)$	$\mathbb{R}$
+	–			$(-\infty, -p)$	$\mathbb{R}$
–	+			$(-p, \infty)$	$\mathbb{R}$
–	–			$(-\infty, -p)$	$\mathbb{R}$

EXAMPLE 38 Sketch the graph $y = \log_3(x-2) + 3$.

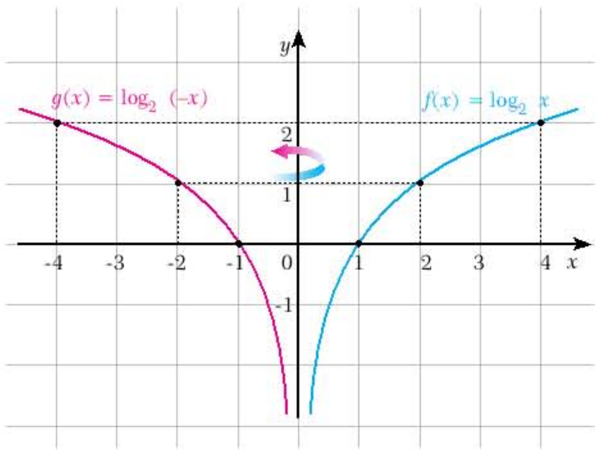
Solution Let $g(x) = \log_3(x-2)$. We can draw the graph of $g(x)$ by shifting the graph of the basic function $f(x) = \log_3 x$ two units to the right (identifying $p = -2 < 0$). Then we shift the graph of $g(x)$ 3 units upward since $y = \log_3(x-2) + 3 = g(x) + 3$ and $k = 3 > 0$.

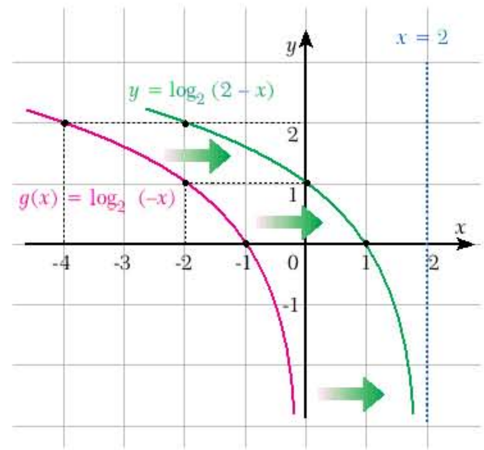




EXAMPLE 39 Sketch the graph $y = \log_2(2-x)$ and determine its domain, range and asymptote.

Solution 1 We begin by sketching the basic function $f(x) = \log_2 x$, then reflect it in the y-axis as the graph of $g(x) = \log_2(-x)$. Since $h(x) = \log_2(2-x) = \log_2|-(x-2)| = g(x-2)$, we sketch $y = g(x-2) = \log_2(2-x)$ by shifting the graph of $g(x)$ two units to the right.



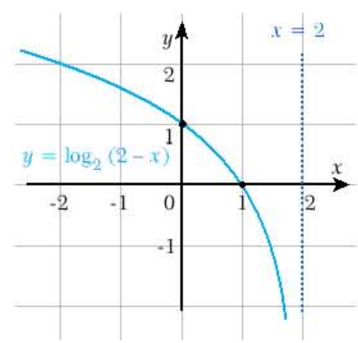


As we can see in the final graph, the domain of $h(x) = \log_2(2-x)$ is $(-\infty, 2)$, the range is $\mathbb{R}$, and the graph has a vertical asymptote at $x = 2$.

Solution 2 We can write the function as $y = \log_2|-(x-2)|$ and then identify $a = 2$, $c = 1$, $d = -1$, $p = -2$ and $k = 0$. So two points on the graph are

$$\left(\frac{1}{d} - p,\ k\right) = \left(\frac{1}{-1} - (-2),\ 0\right) = (1,\ 0) \text{ and}$$

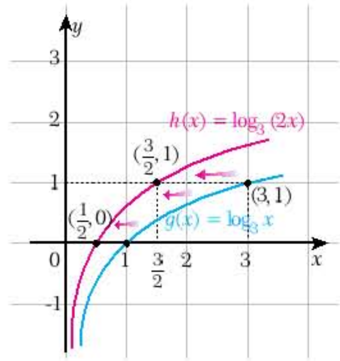
$$\left(\frac{a}{d} - p,\ c + k\right) = \left(\frac{2}{-1} - (-2),\ 1 + 0\right) = (0,\ 1).$$

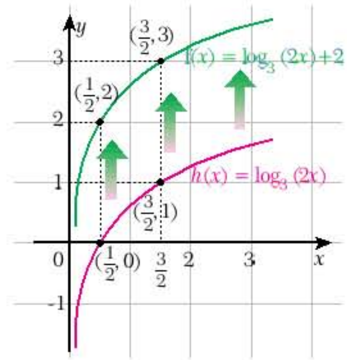


Using $x = -p = 2$ as the vertical asymptote, we draw the graph opposite. The domain of the function is $(-\infty, 2)$, the range is $\mathbb{R}$, and the graph has a vertical asymptote at $x = 2$.

EXAMPLE 40 Sketch the graph of $f(x) = \log_3(2x) + 2$.

Solution After sketching the graph of $g(x) = \log_3 x$, we shrink it by a factor of $\frac{1}{2}$ to get the graph of $h(x) = \log_3(2x)$ (can you see why?). Then we apply a vertical shift 2 units upward to obtain the graph of $f(x) = \log_3(2x) + 2$.



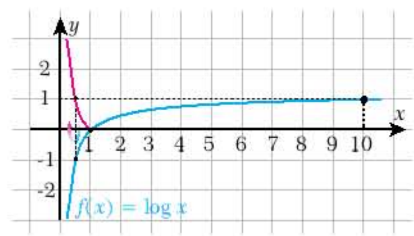


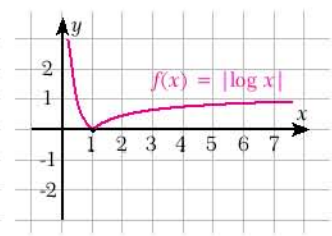
EXAMPLE 41 Sketch each graph.

a. $y = |\log x|$

b. $y = |\log_2 x + 1|$

Solution a. We begin by sketching the graph of $f(x) = \log x$, then we reflect the part of the curve below the x-axis in the x-axis. The result is the graph $y = |\log x|$.

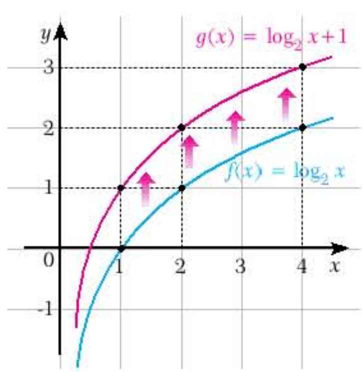


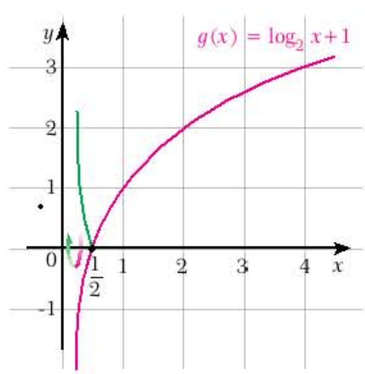


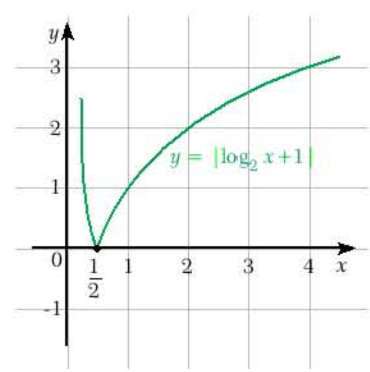
b. Step 1: Graph $f(x) = \log_2 x$.

Step 2: Shift the graph 1 unit upward to obtain $g(x) = \log_2 x + 1$.

Step 3: Reflect the part of the curve below the x-axis in the x-axis to get $y = |\log_2 x + 1|$.





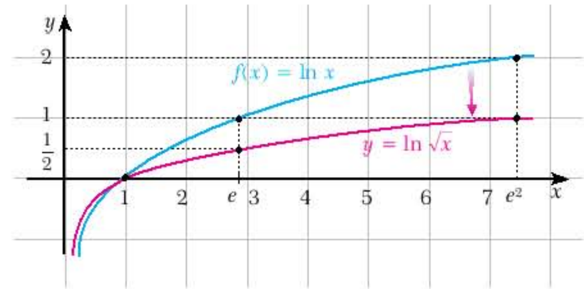


EXAMPLE 42 Sketch the graph $y = \ln \sqrt{x}$.

Solution Since $y = \ln \sqrt{x}$ is equivalent to $y = \frac{1}{2} \cdot \ln x$, we can obtain the graph by shrinking the graph of $f(x) = \ln x$ towards the x-axis.

Since $c = \frac{1}{2}$ and the points $(1, 0)$, $(e, 1)$ and $(e^2, 2)$ are all on the graph of $f(x) = \ln x$, the points $(1, 0 \cdot \frac{1}{2})$, $(e, 1 \cdot \frac{1}{2})$ and $(e^2, 2 \cdot \frac{1}{2})$ will be on the graph of $f(x) = \frac{1}{2}\ln x$.

By plotting these points and joining them with a smooth curve, we obtain the graph shown opposite.



Check Yourself 8

1. Sketch each graph by translating the graph of a simple logarithmic function.

   a. $y = \log_2 x - 3$ b. $y = 2 + \log_{\frac{1}{3}} x$

2. Sketch each graph.

   a. $y = \log_{\frac{1}{3}}(x - 2)$ b. $y = \log(x + 2)$

3. Graph each function and determine its domain, range and asymptote.

   a. $f(x) = \log_2(-x)$ b. $f(x) = -\log_{\frac{1}{3}} x$ c. $f(x) = -|\log x|$ d. $f(x) = \ln |x - 1|$

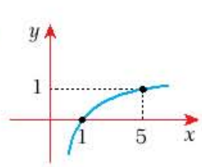
4. Sketch each graph.

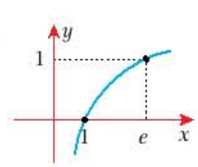
   a. $y = \frac{3}{2}\log_3 x$ b. $y = \log_2 x^3$ c. $y = \log x^2 - 1$

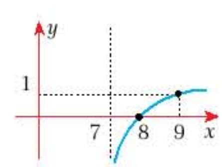
5. Graph each function.

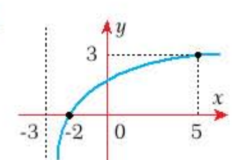
   a. $f(x) = \log_3 \frac{x}{2}$ b. $f(x) = \log_{\frac{1}{4}}(2x)$ c. $f(x) = 2 \ln x - 1$ d. $f(x) = 3 + \log(-2x)$

6. Write the equation of the logarithmic function shown in each graph.

a. 

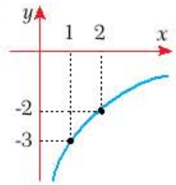
b. 

c. 

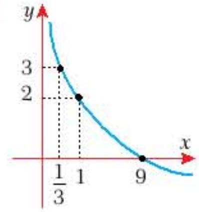
d. 

Answers

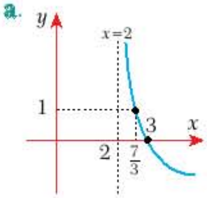
1. a.



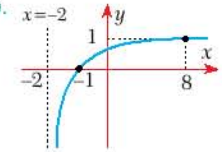
b.



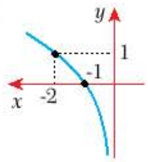
2. a.



b.

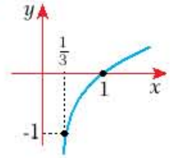


3. a.



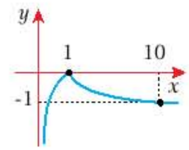
domain : $(-\infty, 0)$
range : $\mathbb{R}$
asymptote : $x = 0$

b.



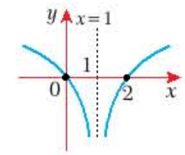
domain : $(0, \infty)$
range : $\mathbb{R}$
asymptote : $x = 0$

c.



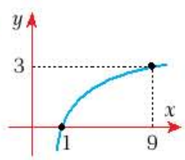
domain : $(0, \infty)$
range : $(-\infty, 0)$
asymptote : $x = 0$

d.

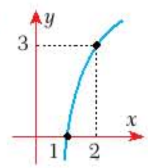


domain : $\mathbb{R} - \{1\}$
range : $\mathbb{R}$
asymptote : $x = 1$

4. a.



b.



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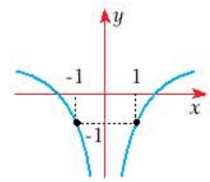
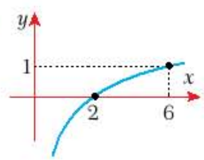
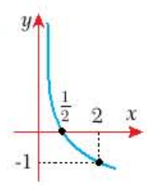


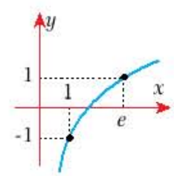
5. a.



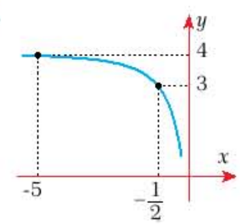
b.



c.



d.



6. a. $y = \log_5 x$ b. $y = \ln x$ c. $y = \log_2 (x - 7)$ d. $y = \log_2 (x + 3)$

4 APPLICATIONS OF LOGARITHMIC FUNCTIONS

The concept of logarithm is not just an abstract mathematical idea. It has many practical uses. In the past, before the invention of scientific calculators, mathematicians performed complicated calculations by first reducing large numbers to logarithms, and then referring to a table of common logarithms. Although today we rarely use logarithms in this way, they still have many other practical uses in the modern world.

Let us look at some examples.

Richter magnitude	Earthquake effect
< 3.5	Earthquake recorded but not felt.
3.5-5.4	Rarely causes damage, but felt, especially on the upper floors of buildings.
5.5-6.0	Slight damage possible to well-designed buildings but can cause major damage to poorly-constructed buildings near the epicenter.
6.1-6.9	Destructive in areas up to 60 miles from the epicenter.
7.0-7.9	Major earthquake causing serious damage.
> 8.0	A 'great earthquake', which can cause severe damage in areas over hundreds of miles.

A. THE RICHTER SCALE

The Richter scale is a scale which is used to measure ground movement. It is commonly used to measure the strength of an earthquake: a higher measurement on the Richter scale means a more violent earthquake. The scale is actually a mathematical formula developed in 1935 by the American geologist Charles Richter. The Richter number R of ground movement is given by the formula

$$R = \log^{n} \frac{I}{I_0}$$

where I is the intensity of the earthquake and I_0 is the minimum intensity that can be felt (called the reference intensity). Intensity is a measure of the shaking and damage caused by the earthquake, and this value changes from location to location.

EXAMPLE 43 The Chilean earthquake of 1960 measured 9.5 on the Richter scale. Compare its intensity with the intensity of the Marmara earthquake of 1999, which measured 7.4 on the Richter scale.

Solution Let I_1 and I_2 be the intensities of the Chilean and Marmara earthquakes, respectively. Then by the formula we have

$$9.5 = \log^{n} \frac{I_1}{I_0} \quad \text{and} \quad 7.4 = \log \frac{I_2}{I_0}.$$

Consequently,

$$\frac{I_1}{I_0} = 10^{9.5} \text{ and } \frac{I_2}{I_0} = 10^{7.4} \Leftrightarrow \frac{I_1}{I_2} = \frac{10^{9.5}}{10^{7.4}} = 10^{2.1} \approx 126.$$

So the Chilean earthquake was 126 times more intense then the Marmara earthquake.

	Location	Date	Magnitude
1.	Chile	1960	9.5
2.	Alaska	1964	9.2
3.	Andreanof Islands, Aleutian Islands	1957	9.1
4.	Kamchatka	1952	9.0
5.	Off western coast of Sumatra, Indonesia	2004	8.8
6.	Off the coast of Ecuador	1906	8.7
7.	Rat Islands, Aleutian Islands	1955	8.7
8.	Northern Sumatra, Indonesia	2005	8.7
9.	Indo-Chinese border	1950	8.6
10.	Kamchatka	1923	8.5

10 biggest earthquakes of the last century

Remark

An earthquake of magnitude 7 on the Richter scale is ten times stronger than an earthquake of magnitude 6.

B. THE pH SCALE

Substance	pH
Battery acid	<1.0
Gastric acid	2.0
Lemon juice	2.4
Cola	2.5
Vinegar	2.9
Orange or apple juice	3.5
Beer	4.5
Coffee	5.0
Tea	5.5
Acid rain	<5.6
Human saliva in cancer patients	4.5-5.7
Milk	6.5
Pure water	7.0
Human saliva	6.5-7.4
Blood	7.34-7.45
Sea water	8.0
Hand soap	8.0-10.0
Household ammonia	11.5
Bleach	12.5
Household lye	13.5

The pH (potential of hydrogen) scale is used in chemistry to determine the acidity or basicity of a solution. Solutions that are not very acidic are called basic. The pH scale has values ranging from zero (the most acidic) to 14 (the most basic).

As you can see from the table on the left, pure water has a pH value of 7. This value is considered neutral: it is neither acidic nor basic. The pH of pure rain is between 5.0 and 5.5, which is slightly acidic. However, when pure rain is combined with sulfur dioxide or nitrogen oxides produced by power plants and automobiles, the rain becomes much more acidic. Typical acid rain has a pH of 4.0.

A decrease of 1 in pH value means that the acidity of the solution becomes ten times greater. For example, cola (pH 2.5) is ten times more acidic than orange or apple juice (pH 3.5).

The pH scale is actually a logarithm of the form

$$\text{pH} = -\log\left[H^+\right]$$

where $\left[H^+\right]$ is the concentration of hydrogen ions in an aqueous solution in moles per liter of the solution.

EXAMPLE 44 A solution of hydrochloric acid is 0.2 molar. Find its pH.

It is simpler to use numbers (pH 4) than exponents ($[H^+] = 10^{-4}$ M) to describe acidity.

Solution Using the formula, pH = –log 0.2 = –(–0.698) = 0.7.

EXAMPLE 45 Find the hydrogen concentration in beer if the pH of beer is 4.82.

Solution $4.82 = -\log |H^+|$

$|H^+| = 10^{-4.82} = 1.51 \cdot 10^{-5}$ mol/L

pH < 7.0	acidic
pH = 7.0	neutral
pH > 7.0	basic

EXAMPLE 46 Calculate the pH of a lemon juice solution which is $5 \cdot 10^{-3}$ molar.

Solution $\text{pH} = -\log(5 \cdot 10^{-3}) = 3 - \log 5 \approx 2.3$

C. THE DECIBEL SCALE

The intensity levels of sounds that we can hear vary from very loud to very soft. Here are some examples of the decibel levels of some common sounds.

Source of sound	dB
Jet takeoff	140
Jackhammer	130
Rock concert	120
Subway	100
Heavy traffic	80
Ordinary traffic	70
Normal conversation	50
Whisper	30
Rustling leaves	10-20
Threshold of hearing	0

The human ear is sensitive to an extremely wide range of sound intensities. The loudest sound a healthy person can hear without damage to the eardrum has an intensity 1 trillion (10^{12}) times the intensity of the softest sound a person can hear.

Sound level is measured in decibels. The sound level β of a sound of intensity I (measured in watts per square meter) is defined by

$$\beta = 10\log\frac{I}{I_0} \text{ decibels,}$$

where $I_0 = 10^{-12}$ watts per square meter is the least intense sound that a human ear can detect.

EXAMPLE 47 Find the sound level of a jet engine during takeoff if the sound intensity is 100 W/m^2.

Solution Applying the sound level formula,

$$\beta = 10\log\frac{100}{10^{-12}} = 10\log 10^{14} = 140 \text{ dB}.$$

So the sound level is 140 dB.

EXAMPLE 48 Determine the sound level of ordinary traffic if it is known that this sound is 100 times more intense than the sound level of normal conversation (50 dB).

Solution The sound level of normal conversation is 50 decibels, so we use the sound level formula

$50 = 10\log\frac{I}{I_0}$ and get $\log\frac{I}{I_0} = 5$.

A sound 100 times as intense as I has sound level $100I$. Thus the loudness of ordinary traffic is $\beta = 10\log\frac{100I}{I_0} = 10(\log 100 + \log\frac{I}{I_0}) = 10(2 + \log\frac{I}{I_0}) = 10 \cdot (2 + 5) = 70$ dB.

Check Yourself 9

1. Compare the intensities of the Mexican earthquake of 1985 (7.8 on the Richter scale) and the San Francisco earthquake of 1989 (7.1 on the Richter scale).
2. Find the hydrogen ion concentration $|H^+|$ of milk, given that its pH is 6.3.
3. The sound level of a moving subway train measured 98 dB. Find its intensity in W/m^2.

Answers

1. the Mexican earthquake was 5 times stronger 2. $5 \cdot 10^{-7}$ mol/L 3. $6 \cdot 10^{-3}$ W/m^2

EXERCISES 2.3

A. The Richter Scale

1. A scientist measured the intensity of an earthquake to be 120,000 times the reference intensity (minimum intensity). The scientist needs to report a Richter scale reading to a newspaper reporter. Which number should he give to the reporter?

2. The San Francisco earthquake of 1989 measured 7.1 on the Richter scale. Compare this with the earthquake in Indonesia in 1985 which measured 6.8.

3. If one earthquake is 30 times more intense than another, how much larger is its measurement on the Richter scale?

4. The Alaskan earthquake of 1964 had a magnitude of 8.6 on the Richter scale. How many times more intense was this than the 1999 Marmara earthquake which measured 7.4 on the scale?

5. Use the Richter scale formula to find the magnitude of an earthquake with the given intensity.

   a. 100 times the intensity of I_0

   b. 10,000 times the intensity of I_0

   c. 100,000 times the intensity of I_0

B. The pH Scale

6. What is the pH of a solution whose concentration of hydrogen ions is $6.5 \cdot 10^{-8}$ moles per liter?

7. The most acidic rainfall ever measured occurred in Scotland in 1974. Its pH was 2.4. Find the hydrogen ion concentration of this rain.

8. Most solutions have a pH value between 1 and 14. Find the corresponding range of $|H^+|$.

C. The Decibel Scale

9. Find the loudness of a dishwasher that operates at an intensity of 10^{-5} W/m^2.

10. The intensity of a sound A is 50 times the intensity of a sound B. What is the difference between the two sound levels in decibels?

11. A man hears a noise which has an intensity of $10^{-9.8}$. What is the sound level of the noise in decibels?

ESTIMATING WORLD POPULATION

Exponential and logarithmic functions are often used by mathematicians to model changes in the real world. One example of such a change is the growth and decline of world population. Mathematicians can use a model to predict the population of our planet at a certain time in the future, or to estimate how different events in history might have affected world population today.

Since the second century, the population of the world has been affected by outbreaks of a disease known as the *bubonic plague*. During the fourteenth and fifteenth centuries (1348-1405), the greatest bubonic plagues (known as the Black Death) spread in six waves from central Asia through India and North Africa to Europe and beyond. In these regions, approximately one-third of the population died from the disease. The plague caused a decrease in world population from 470 million people in 1348 to 370 million in 1400.

Estimated World Population	
Year	Population
1200	0.35×10^9
1300	0.38×10^9
1348	0.47×10^9
1400	0.37×10^9
1500	0.45×10^9
1600	0.49×10^9
1700	no accurate estimate available
1800	0.91×10^9
1900	1.60×10^9
2000	6.05×10^9

We can use math to investigate what would have happened to world population if the Black Death had not occurred. How many people would there be on Earth now if nobody had died from the disease?

The table on the left shows the estimated world population in the year beginning each century. We can use this information to create a formula which models the population growth in each century. The formula is $P = P_0 \cdot e^{rt}$ where t is the duration of the period, P_0 is the initial population of the period, P is the terminal population of the period and r is the population growth rate. Rewriting this formula in terms of r gives us $r = \frac{1}{t} \cdot \ln \frac{P}{P_0}$ which is the formula for the population growth rate. Using this formula with periods of 100 years gives us the table on the right.

Population Growth Rates	
Interval	r
1200-1300	0.82×10^{-3}
1300-1348	4.40×10^{-3}
1400-1500	1.96×10^{-3}
1500-1600	0.85×10^{-3}
1600-1800	3.10×10^{-3}
1800-1900	5.64×10^{-3}
1900-2000	13.30×10^{-3}

We can use these growth rates and the population figure for 1348 to predict the world population now if the Black Death had not occurred. To do this, we assume that all the other events between 1348 and 2000 were the same.

Predicted world population without Black Death	
Year	Predicted Population
1400	0.59×10^9
1500	0.72×10^9
1600	0.78×10^9
1800	1.45×10^9
1900	2.50×10^9
2000	9.45×10^9

As we can see from the table, if the Black Death had not occurred then in the year 2000 there would have been 9.45 billion people, or twice the world's current population, living on the planet.

CHAPTER 2 SUMMARY

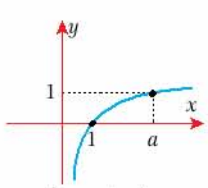
1. Logarithms

- The logarithm of a number N to a base a is the power to which a must be raised in order to obtain N. More formally, for $a > 0$, $a \neq 1$, and $x > 0$, the real number y defined by $y = \log_a x \Leftrightarrow x = a^y$ is called the logarithm of x to base a.
- $a^{\log_a x} = x$ is called the fundamental identity of logarithms.
- Logarithms are not defined for negative numbers and zero.
- Logarithms to base 10 are called common logarithms. We write $\log x$ to mean the common logarithm of x.
- Logarithms to the base e are called natural logarithms. We write $\ln x$ to mean the natural logarithm of x.
- Properties of Logarithms

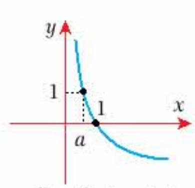
1. $\log_a a = 1$	2. $\log_a 1 = 0$
3. $\log_a(x \cdot y) = \log_a x + \log_a y$	4. $\log_a x^m = m \cdot \log_a x$
4a. $\log_a \dfrac{1}{x^n} = -n \cdot \log_a x$	4b. $\log_a \sqrt[m]{x^n} = \dfrac{n}{m} \cdot \log_a x$
5. $\log_a(\dfrac{x}{y}) = \log_a x - \log_a y$	6. $\log_{a^n} x^m = \dfrac{m}{n} \cdot \log_a x$
7. $\log_a x = \dfrac{\log_b x}{\log_b a}$	7a. $\log_a b = \dfrac{1}{\log_b a}$
7b. $\log_a b \cdot \log_b a = 1$	8. $b^{\log_a c} = c^{\log_a b}$

2. Logarithmic Functions

- For $a > 0$, $a \neq 1$, a function $f: (0, \infty) \to \mathbb{R}$ defined by $f(x) = \log_a x$ is called a basic logarithmic function with base a.
- The graph of a basic logarithmic has one of the following general forms:



when $a > 1$



when $0 < a < 1$

- Properties of the Basic Logarithmic Function $f(x) = \log_a x$

1. The domain is is the set of positive real numbers.
2. The range is the set of all real numbers.
3. If $a > 1$ then the function is strictly increasing.
4. If $0 < a < 1$ then the function is strictly decreasing.
5. The y-axis ($x = 0$) is a vertical asymptote of the graph of f.
6. The graph of f has no y-intercept, and the x-intercept is $(1, 0)$.
7. $(1, 0)$ and $(a, 1)$ are two points on the graph of f.

- If both a and x are from the same interval $(0, 1)$ or $(1, \infty)$ then $\log_a x > 0$, otherwise $\log_a x < 0$.
- Logarithmic functions are bijective. Logarithmic and exponential functions are inverse functions.
- $\log_a x = \log_a y \Leftrightarrow x = y$

3. Simple Variations of Logarithmic Functions

- A function of the form $f(x) = c \cdot \log_a[d(x + p)] + k$ where a, c, p, d and k are real numbers with $c, d \neq 0$ is called a logarithmic function with base a.
- The graph of a logarithmic function can be obtained by transforming the graph of a basic logarithmic function $g(x) = \log_a x$.
- For a function $g(x) = c \cdot \log_a[d(x + p)] + k$, the different constants have the following effect on the graph of $f(x) = \log_a x$:

  – k represents a vertical shift.

  – p represents a horizontal shift.

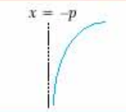
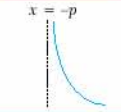
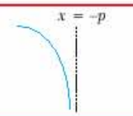
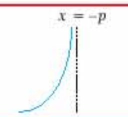
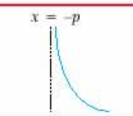
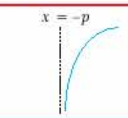
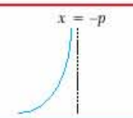
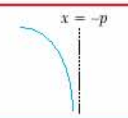
  – c represents a vertical shrink or stretch.

  – d represents a horizontal shrink or stretch.

- Any point (x, y) on the graph of $f(x)$ moves to $(\dfrac{x}{d} - p,\ c \cdot y + k)$ for $y = c \cdot \log_a[d \cdot (x + p)] + k$.
- Properties of the General Logarithmic Function $f(x) = c \cdot \log_a[d \cdot (x + p)] + k$

1. The range is $\mathbb{R}$.
2. $x = -p$ is a vertical asymptote.
3. If $c < 0$, a reflection in the y-axis occurs and the domain of f becomes $(-\infty, -p)$, otherwise the domain is $(-p, \infty)$.
4. $(\dfrac{1}{d} - p,\ k)$ and $(\dfrac{a}{d} - p,\ c + k)$ are two points on the graph of f.

- The following table shows how the values of c and d affect the behavior of $f(x) = c \cdot \log_a[d \cdot (x + p)] + k$.

c	d	$a > 1$	$0 < a < 1$
+	+		
+	–		
–	+		
–	–		

4. Applications of Logarithmic Functions

- In the past, before the invention of scientific calculators, mathematicians reduced large numbers to logarithms and then performed calculations by using a table of logarithms.
- The Richter scale is used to measure the strength of ground movement. It is defined by the formula $R = \log \frac{I}{I_0}$, where R is the number on the scale, I is the intensity of the movement, and I_0 is the minimum intensity that can be felt.
- The pH scale is used to determine the acidity of a solution. It is defined by a logarithm of the form pH $= -\log[H^+]$, where $[H^+]$ is the concentration of hydrogen ions in a solution in moles per liter.
- The decibel scale measures sound levels in decibels. A sound level β is defined by $\beta = 10 \cdot \log \frac{I}{I_0} dB$, where I is the intensity of the sound and $I_0 = 10^{-12}$ W/m^2 is the least intense sound that a human ear can detect.

Concept Check

1. What is a logarithm?
2. Why are logarithms useful? Give two examples.
3. What are the existence conditions for logarithms?
4. What is the fundamental identity of logarithms?
5. Name two special types of logarithm.
6. Is it true that $\log_a (x \cdot y) = \log_a x + \log_a y$ for all real values of x and y?
7. How can we evaluate a logarithm to a base other than 10 using a scientific calculator?
8. State the Change of Base formula properties for logarithms.
9. Define the basic and general form of a logarithmic function.
10. How does the base of a basic logarithmic function affect its graph?
11. What kind of asymptote do logarithmic functions have?
12. Describe the monotony of basic logarithmic functions.
13. When does a basic logarithmic function have negative values?
14. When does a basic logarithmic function have values greater than 1?
15. State the common properties of basic and general log-arithmic functions.
16. How do the values of c and d affect the graph of a general logarithmic function of the form $f(x) = c \cdot \log_a[d(x + p)] + k$? Describe the effect of the other constants.
17. To which point will (x_0, y_0) on the graph of $f(x) = \log_a x$ move on the graph of $y = 2 \cdot \log_a(5x - 1) - 3$?
18. How do we measure the magnitude of an earthquake? State a formula.
19. How do we measure the acidity of solutions?
20. Describe one more practical application of logarithms.

CHAPTER REVIEW TEST 2A

1. What is the inverse of the logarithmic function $f(x) = \log_2(x + 1)$?

A) $f^{-1}(x) = 2^x - 1$ B) $f^{-1}(x) = 2^{x-1}$
C) $f^{-1}(x) = 2^x + 1$ D) $f^{-1}(x) = 2^{x+1}$
E) $f^{-1}(x) = 2^x - 2$

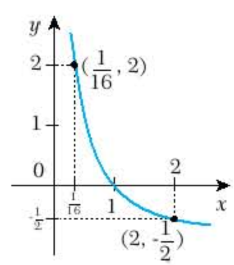
2. Which of the following points is on $y = 2 + \log_3 x$?

A) (2, 1) B) (2, 3) C) (3, 1)
D) (3, 3) E) (9, 3)

3. What is the largest possible domain of $f(x) = \log_4(2x + 1)$?

A) $(-\frac{1}{2}, -\frac{1}{4})$ B) $(-\frac{1}{4}, \infty)$ C) $(-\frac{1}{2}, \infty)$
D) $(\frac{1}{2}, \infty)$ E) $(\frac{1}{4}, \infty)$

4. Which function has the graph shown opposite?



A) $f(x) = \log_{\frac{1}{16}} x$
B) $f(x) = \log_{\frac{1}{4}} x$
C) $f(x) = \log_{\frac{1}{2}} x$
D) $f(x) = \log_2 x$
E) $f(x) = \log_4 x$

5. Calculate $\log_2 32$.

A) 1 B) 2 C) 4 D) 5 E) 6

6. Solve $\log_4 x = \frac{1}{4}$ for x.

A) $\sqrt{2}$ B) $\sqrt[3]{4}$ C) $\sqrt[4]{2}$ D) 2 E) 6

7. $\log_x 81 = -4$ is given. What is x?

A) $\frac{1}{9}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) 3 E) 9

8. What is the logarithm of 125 to the base 5?

A) 5 B) 3 C) 2 D) 1 E) $\frac{1}{3}$

9. Which of the following statements is/are true for the logarithmic function $f(x) = \log_3 x$?

I. $f(x)$ is increasing for all $x \in \mathbb{R}$.
II. $f(x)$ has a range of $\mathbb{R}^+$.
III. $f(1) = 0$
IV. $f(3) = 3$

A) I and II B) II and III C) I and III
D) only III E) only II

10. What is the largest possible domain of $f(x) = \log_x(5x - 7)$?

A) $(1, \infty)$ B) $(0, \frac{7}{5}]$ C) $(0, \frac{7}{5})$
D) $(\frac{7}{5}, \infty)$ E) $(1, \frac{7}{5})$

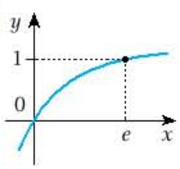
11. Calculate $\log \dfrac{1}{1000}$.

A) -4 B) -3 C) -2 D) 0.001 E) 2

12. Given that $f(x) = \log(2x - 1)$, what is $f^{-1}(2)$?

A) $\dfrac{103}{2}$ B) 51 C) $\dfrac{101}{2}$ D) 50 E) $\dfrac{99}{2}$

13. Which graph shows $y = \ln x$?

A)


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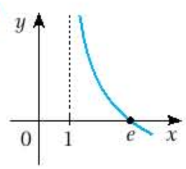
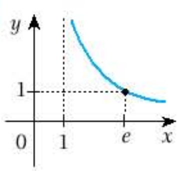
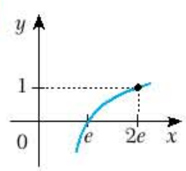
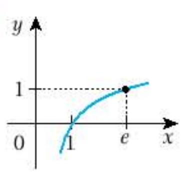


C)


D)


E)


14. Given $f(x) = \ln x$ and $(g \circ f)(x) = x$, find $g(x)$.

A) 10^x B) $\dfrac{1}{e^x}$ C) $\dfrac{x}{\ln x}$ D) e^x E) $e^x + 1$

15. Evaluate $\log 1000 - \ln e^2 - \log_4 64$.

A) -3 B) -2 C) -1 D) 2 E) 3

16. $\log_8 a = \dfrac{4}{3}$ is given. Find a.

A) 4 B) 8 C) 16 D) 2^8 E) 2^{16}

17. Which expression is equal to $\log \dfrac{x \cdot y^2}{z^3}$?

A) $\log x + 2\log y + 3\log z$

B) $\log x + 2\log y - 3\log z$

C) $\log(x + 2y - 3z)$

D) $\log x + 2\log y - \dfrac{1}{3}\log z$

E) $\log x + 2\log y - \log \dfrac{z}{3}$

18. $\log 3 = a$, $\log 5 = b$ and $\log 210 = c$ are given. Write $\log 7$ in terms of a, b and c.

A) $c - a - b$ B) $c - a - 1$

C) $c - a - b + 1$ D) $c - a - b - 1$

E) $c - a + 1$

19. Which expression is equal to $\ln \dfrac{1}{\sqrt{3}}$?

A) $-\dfrac{1}{3}\ln 3$ B) $-\dfrac{1}{2}\ln 3$ C) $\dfrac{1}{3}\ln 3$

D) $-\dfrac{1}{2}(\ln 3 - 1)$ E) $\dfrac{1}{2}(\ln 3 - 1)$

20. Evaluate $\dfrac{1}{\log_{12} 6} - \dfrac{1}{\log_8 6} + \dfrac{1}{\log_4 6}$.

A) $\log_6 3$ B) $\log_3 6$ C) 0 D) 1 E) 2

CHAPTER REVIEW TEST 2B

1. Evaluate $\log_{2\sqrt{2}} 2\sqrt[3]{2}$.

A) $\dfrac{6}{7}$ B) $\dfrac{7}{8}$ C) $\dfrac{8}{9}$ D) $\dfrac{9}{8}$ E) $\dfrac{8}{7}$

2. $\log 2 = a$ and $\log 3 = b$ are given. What is $\log_6 15$ in terms of a and b?

A) $\dfrac{b-a+1}{a+b}$ B) $\dfrac{a-b+1}{a+b}$ C) $\dfrac{a-b-1}{a+1}$

D) $\dfrac{a-b-1}{a+2}$ E) $\dfrac{a+b}{a+b+1}$

3. A triangle ABC has sides $a = \log 4$, $b = \log 20$ and $c = \log 125$. What is its perimeter?

A) 4 B) 5 C) 6 D) 7 E) 8

4. Evaluate $\log_8 16 - \log_9 27 + \log\sqrt{10} - \ln\sqrt[4]{e}$.

A) $\dfrac{1}{4}$ B) $\dfrac{1}{6}$ C) $\dfrac{1}{8}$ D) $\dfrac{1}{9}$ E) $\dfrac{1}{12}$

5. Which expression is equal to $\ln x + \ln y - \ln z$?

A) $\ln(x + y - z)$ B) $\ln\dfrac{x+y}{z}$ C) $\ln\dfrac{xy}{z}$

D) $\ln(xy - z)$ E) $\dfrac{\ln(x+y)}{\ln z}$

6. Evaluate $\log_6 9 + \log_6 12 + \log_6 2$.

A) 3 B) 6 C) 9 D) 10 E) 12

7. Calculate 9^a if $a = \log_3 5$.

A) 81 B) 25 C) 15 D) 9 E) 13

8. Evaluate $\log_7 8 \cdot \log_8 7 \cdot \log_7 10$.

A) $\log 7$ B) $\ln 7$ C) $\ln 10$ D) $\log_7 10$ E) 1

9. Write $2\log a + \log b - \log(a + 2b)$ as a single logarithm.

A) $\log\dfrac{a^2+b}{a+2b}$ B) $\log\dfrac{a}{2}$ C) $\log\dfrac{2a+b}{a+b}$

D) $\log(a-b)$ E) $\log\dfrac{a^2b}{a+2b}$

10. What is the common logarithm of $\dfrac{(0.4)^2}{20^4}$?

A) -2 B) -4 C) -6 D) -8 E) -10

11. Calculate log 50 if log 2 = 0.301.

A) 1.701 B) 1.699 C) 1.30 D) 0.699 E) 0.602

12. $a = 13.72$ and $b = 13720$ are given. What is $\log a - \log b$?

A) –5 B) –4 C) –3 D) –2 E) 0.01

13. Evaluate $\dfrac{1}{\log_{24} 30} + \dfrac{1}{\log_{225} 30} - \dfrac{1}{\log_{6} 30}$.

A) $\dfrac{1}{3}$ B) $\dfrac{1}{2}$ C) 1 D) 2 E) 3

14. Evaluate $(a^4)^{\log_{a^2} \sqrt{3}}$.

A) $\sqrt[3]{3}$ B) $\sqrt{3}$ C) 3 D) $3\sqrt{3}$ E) 9

15. If $\log_{\frac{m}{n}} n = x$, what is $\log_{\frac{m}{n}} m$ in terms of x?

A) $\dfrac{1}{x-1}$ B) $\dfrac{x}{x+1}$ C) $\dfrac{x-1}{x}$

D) $-x - 1$ E) $x + 1$

16. $a = \log_2 5$, $b = \log_5 4$ and $c = \log_3 8$ are given. Which statement is true?

A) $b < a < c$ B) $c < a < b$ C) $b < c < a$

D) $c < b < a$ E) $a < c < b$

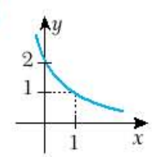
17. Evaluate $\log_{16} 27 \cdot \log_{125} 32 \cdot \log_9 625$.

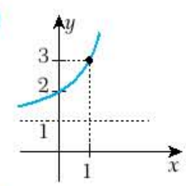
A) $\dfrac{5}{2}$ B) $\dfrac{5}{3}$ C) $\dfrac{4}{3}$ D) $\dfrac{2}{3}$ E) $\dfrac{3}{5}$

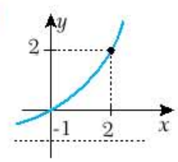
18. How many integer values of a satisfy the existence conditions for $\log_{7-a}(a + 9)$?

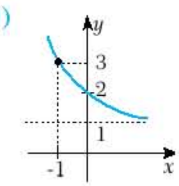
A) 17 B) 16 C) 15 D) 14 E) 13

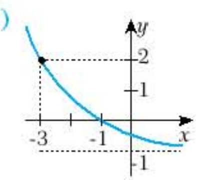
19. Which figure shows the graph $y = 2^{-x} + 1$?

A)


B)


C)


D)


E)


20. What is the inverse of $f(x) = (0.2)^{-x} + 1$?

A) $f^{-1}(x) = 1 + \log_5 x$ B) $f^{-1}(x) = 1 + \log_x 5$

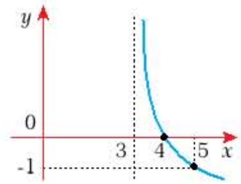
C) $f^{-1}(x) = \log_5(x - 1)$ D) $f^{-1}(x) = \log_5(x + 1)$

E) $f^{-1}(x) = -1 + \log_5 x$

1. If $\log_a b = \log_b c = \log_c a$, what is $\log_a c + \log_c a + \log_b a$?

A) $\frac{9}{2}$ B) 4 C) 3 D) 2 E) $\frac{3}{2}$

2. Which function could have the graph in the figure?



A) $f(x) = \log_{\frac{1}{3}}(x-3)$

B) $f(x) = \log_3(x-3)$

C) $f(x) = \log_{\frac{1}{2}}(x-3)$

D) $f(x) = \log_2(x-3)$

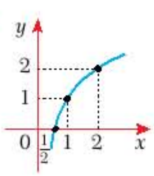
E) $f(x) = -3 + \log_{\frac{1}{3}} x$

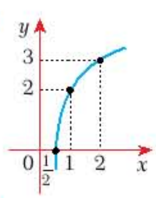
3. $p = \log_2 9$, $q = \log_3 83$ and $r = \log_5 123$ are given. Which statement is true?

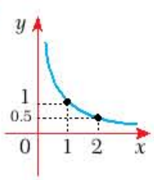
A) $p > r > q$ B) $q > p > r$ C) $r > q > p$

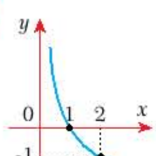
D) $r > p > q$ E) $q > r > p$

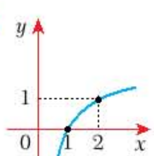
4. Which figure shows the graph $y = 1 + \log_2 x$?

A)


B)


C)


D)


E)


5. Given $x = \log_2 3$, calculate $\dfrac{2^{3x} - 2^{-3x}}{2^x - 2^{-x}}$.

A) $1\frac{1}{9}$ B) 3 C) $5\frac{1}{9}$ D) $10\frac{1}{7}$ E) $10\frac{1}{9}$

6. $\log_3 2 = a$ is given. What is $\log_4 6$?

A) $\dfrac{2a}{a-1}$ B) $\dfrac{a+1}{2a}$ C) $\dfrac{a+1}{a-1}$

D) $\dfrac{a-1}{2a}$ E) $\dfrac{2a}{a+1}$

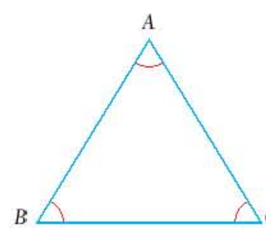
7. In the triangle ABC shown opposite,

$m(\angle A) = (45 \log_6 y^2)°$,

$m(\angle B) = (90 \log_6 x)°$ and

$m(\angle C) = (180 \log_6 \sqrt{z})°$.

If $\log_{x \cdot z} y = -3$, what is y?



A) 36 B) 72 C) 144 D) 216 E) 256

8. Which statement is <u>false</u>?

A) $-1 < \log \dfrac{1}{9{,}9} < 0$ B) $1 < \log 11 < 2$

C) $-3 < \log \dfrac{2}{201} < -2$ D) $-3 < \log 0.07 < -2$

E) $3 < \log 2^{10} < 4$

9. Evaluate $\log \dfrac{100^3 \cdot \sqrt{0.0001}}{0.01^2}$.

A) 7 B) 8 C) 9 D) 10 E) 12

10. If $\log 2 \cong 0.301$, how many digits are there in the number $25^4 \cdot 8^{16}$?

A) 22 B) 21 C) 20 D) 19 E) 18

11. If $\log(a + b) = \log a + \log b$, what is a in terms of b?

A) $\frac{b+1}{b}$ B) $\frac{b}{b-1}$ C) $\frac{b}{b+1}$ D) $\frac{b-1}{b+1}$ E) $\frac{1-b}{b}$

12. $f(x) = 3^{x-2}$ and $g(x) = \log_3(5x - 3)$ are given. What is p if $(f \circ g)(p) = 3$?

A) 3 B) 5 C) 6 D) 9 E) 15

13. If $\log 72 = a$ and $\log 2 = b$, what is $\log 3$ in terms of a and b?

A) $\frac{2a+b}{3}$ B) $\frac{a-3b}{2}$ C) $3a - 2b$ D) $2a + 3b$ E) $\frac{b-2a}{3}$

14. Evaluate $\sqrt{2^{\frac{3}{\log_9 2}}}$.

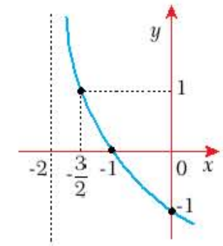
A) 27 B) 18 C) 9 D) 6 E) 3

15. What is the largest possible domain of

$f(x) = \sqrt{2 - \log_3(x+4)}$?

A) $(-4, 3]$ B) $(3, 9]$ C) $(-4, 5]$ D) $(-3, 9]$ E) $(2, 5]$

16. Which function could have the graph in the figure?



A) $f(x) = \log_{\frac{1}{2}}(x+2)$ B) $f(x) = \log_{\frac{1}{2}}(x+1)$

C) $f(x) = \log_{\frac{1}{2}}(x-1)$ D) $f(x) = \log_2(x+2)$

E) $f(x) = \log_{\frac{1}{2}}(x-2)$

17. $f(x) = 3^x$, $(g \circ f)(x) = 2x - 1$ and $g(p) = 3$ are given. What is p?

A) $\frac{1}{3}$ B) $\sqrt{3}$ C) 3 D) 9 E) 27

18. How many digits does x have if

$\log_2(\log_3(\log(5x))) = 1$?

A) 5 B) 6 C) 8 D) 9 E) 10

19. What is $x + y + z$ if $\log_2(\log_3(\log_4 x)) = 0$, $\log_3(\log_4(\log_2 y)) = 0$, and $\log_4(\log_2(\log_3 z)) = 0$?

A) 50 B) 58 C) 71 D) 89 E) 111

20. Calculate $x^2 + y^2$ given $\log_2(x - y) = 5 - \log_2(x + y)$ and $\frac{\log x - \log 4}{\log y - \log 3} = -1$.

A) 40 B) 48 C) 60 D) 74 E) 90

CHAPTER REVIEW TEST 2D

1. Calculate $\log 0.09$ if $\log 3 = 0.4771$.

A) -2.9542 B) -1.9542 C) 0.0458 D) -2.4771 E) -1.0458

2. $\log_2 x = 98$, $\log_3 y = 56$ and $\log_5 z = 42$ are given. Which statement is true?

A) $z < y < x$ B) $z < x < y$ C) $y < z < x$ D) $y < x < z$ E) $x < z < y$

3. $f(x) = 1 + \ln x$, $g(x) = x^2$ and $(f \circ g)(a) = (g \circ f)(a)$ are given. Find a.

A) $\frac{1}{e}$ B) $\sqrt{e}$ C) e D) e^2 E) 1

4. Calculate $\sqrt{25^{\frac{1}{\log_6 5}} + 49^{\frac{1}{\log_8 7}}}$.

A) 7 B) 10 C) 12 D) 14 E) 28

5. Calculate $\log 25$ using $\log 2 = 0.30103$.

A) 0.48856 B) 0.69897 C) 1.29897 D) 1.39794 E) 1.42765

6. What is the sum of the integers in the largest possible domain of $f(x) = \log_{x-1}(7 - x)$?

A) 18 B) 20 C) 23 D) 24 E) 28

7. $\log 5 = x$, $\log 3 = y$ and $\log 2 = z$ are given. Write $\log 1800$ in terms of x, y and z.

A) $x + 2y + 3z$ B) $2x + y + z$ C) $x + 2y + z$ D) $3x + y + 2z$ E) $2x + 2y + 3z$

8. $\frac{1}{2}(\log x + \log y) = \log|\frac{1}{3}(x + y)|$ are given. What is $(x - y)^2$?

A) $2xy$ B) $4xy$ C) $5xy$ D) $6xy$ E) $9xy$

9. How many digits are there in 9^{15} if $\log 3 \cong 0.477$?

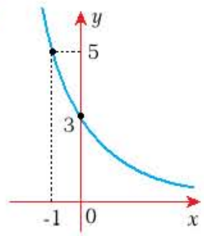
A) 12 B) 13 C) 14 D) 15 E) 16

10. How many natural numbers are there in the largest possible domain of $f(x) = \sqrt[4]{\ln(4 - x)}$?

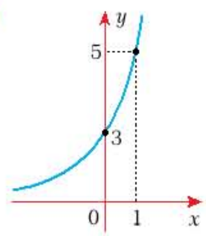
A) 1 B) 2 C) 3 D) 4 E) 5

11. $f(x) = \log_3(x-2)$ is given. Which figure shows the graph of $f^{-1}(x)$?

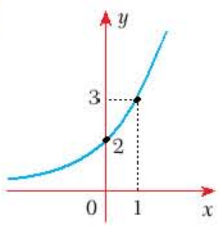
A)



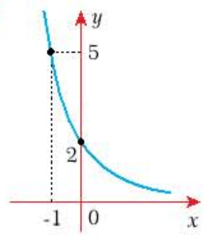
B)



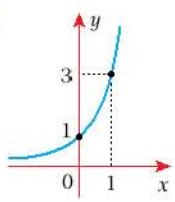
C)



D)



E)



12. Given $\log_7 13 = a$ and $\log_{13} 17 = b$, write $\log_{17} 7$ in terms of a and b.

A) $\dfrac{1}{a \cdot b}$ B) $a + b$ C) $\dfrac{a}{b}$ D) $a \cdot b$ E) $\dfrac{b}{a}$

13. Evaluate $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \ldots \cdot \log_{63} 64$.

A) 2 B) 3 C) 4 D) 5 E) 6

14. $\log_3 29 = a$, $\log_b 29 = 2$ and $\log_{43} c = \dfrac{1}{2}$ are given. Which statement is true?

A) $a < c < b$ B) $b < c < a$ C) $a < b < c$
D) $c < a < b$ E) $c < b < a$

15. $\log_3(a \cdot b) = 7$ and $\log_3 \dfrac{a}{b} = 1$ are given. What is $\log_a b$?

A) $\dfrac{4}{3}$ B) $\dfrac{3}{4}$ C) $\dfrac{2}{3}$ D) $\dfrac{1}{3}$ E) $\dfrac{1}{4}$

16. Evaluate $4^{\log_8 2\sqrt{2}} + \log_{\sqrt{3}-\sqrt{2}}(\sqrt{3}+\sqrt{2})$.

A) 1 B) 2 C) 3 D) 4 E) 5

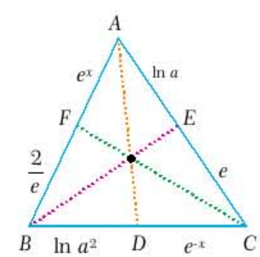
17. If $p(f(x)) = x \cdot f(x+1)$, what is $p(p(\ln x))$?

A) $\ln(x+1) \cdot \ln(x+2)$ B) $x \cdot \ln(x+2)$
C) $(x+1) \cdot \ln(x+2)$ D) $x \cdot \ln(x+2)^{x+1}$
E) $(x+2) \cdot \ln(x+1)^x$

18. Evaluate $\left(b^{\frac{\log_{100} a}{\log a}} \cdot a^{\frac{\log_{100} b}{\log b}}\right)^{2 \cdot \log_{a \cdot b}(a+b)}$.

A) a B) b C) $a + b$ D) $a - b$ E) $a \cdot b$

19. In the triangle ABC shown opposite, AD, BE and CF intersect at a point. Use the measurements in the figure to calculate x.



A) 0 B) $\ln e$ C) $\ln\dfrac{1}{e}$ D) $\ln a$ E) $\ln\dfrac{1}{a}$

20. What is $\sqrt{(\log 2)^2 + (\log \frac{1}{2})^2}$?

A) 0 B) $\log \sqrt{2}$ C) $\sqrt{2} \log 2$
D) $\log(\frac{1}{2})$ E) $\sqrt{2}\log(\frac{1}{2})$

Chapter 3

EXPONENTIAL AND LOGARITHMIC EQUATIONS AND INEQUALITIES

1 EXPONENTIAL EQUATIONS AND INEQUALITIES

A. EXPONENTIAL EQUATIONS

An exponential equation is an equation in which the unknown appears only in the exponent(s). For example, $3^{x+2} = 1$ is an exponential equation, but $x \cdot 3^x = 3$ is not an exponential equation. A solution of an exponential equation is a value of x which satisfies the equation. The set which contains all the solutions of an exponential equation is called the solution set of the equation. Two exponential equations are said to be equivalent if they have exactly the same solution set. Solving an exponential equation means finding its solution set.

There is no single method that we can use to solve all exponential equations. However, we can apply some principles to solve certain types of equation. Let us look at these different types in turn.

If $f(x)$ is one-to-one then $f(x) = f(y) \Leftrightarrow x = y$.
For $f(x) = a^x$,
$a^x = a^y \Leftrightarrow x = y$.

1. Equations of the Form $a^{f(x)} = a^{g(x)}$

Since exponential functions are one-to-one, $a^{f(x)} = a^{g(x)}$ means $f(x) = g(x)$. So the equations $f(x) = g(x)$ and $a^{f(x)} = a^{g(x)}$ are equivalent: they have the same solution set.

EXAMPLE 1 Solve each equation for x.

a. $2^{3x} = 2^9$ b. $7^{x+9} = 7^{2x-15}$ c. $(\frac{3}{11})^{3x-10} = (\frac{11}{3})^{7x-10}$

Solution a. The expressions on each side of the equation have the same base (2). So we can simply equate the exponents:

$$2^{3x} = 2^9 \Leftrightarrow 3x = 9 \Leftrightarrow x = 3.$$

b. Since the bases are the same, the exponents must be equal. So we can write $x + 9 = 2x - 15$, which gives us $x = 24$.

c. First we need to make the bases equal. Since $\frac{3}{11}$ is the multiplicative inverse of $\frac{11}{3}$,

$$(\frac{3}{11})^{3x-10} = (\frac{11}{3})^{7x-10} \Leftrightarrow (\frac{3}{11})^{3x-10} = (\frac{3}{11})^{-(7x-10)} \Leftrightarrow (\frac{3}{11})^{3x-10} = (\frac{3}{11})^{10-7x}.$$

Now we can solve the equation by equating the exponents: $3x - 10 = 10 - 7x \Leftrightarrow 10x = 20 \Leftrightarrow x = 2$.

- y is the multiplicative inverse of x if $x \cdot y = 1$.
- $(\frac{a}{b})^m = (\frac{b}{a})^{-m}$

Check Yourself 1

Solve each equation for x.

a. $2^{5x-3} = 2^{x+1}$ b. $4^{3-2x} = \frac{1}{16}$ c. $3^{2x+1} = 3^{-x^2}$ d. $3^{x/2} \cdot 5^{x/2} = 225$

Answers

a. $\{1\}$ b. $\{\frac{5}{2}\}$ c. $\{-1\}$ d. $\{4\}$

2. Equations of the Form $a^{f(x)} = b$

$a^x = y \Leftrightarrow x = \log_a y$

Notice that the equation $a^{f(x)} = b$ has no solution if $b \le 0$ because exponential numbers of the form $a^{f(x)}$ are strictly positive.

If $b > 0$ then we can solve $a^{f(x)} = b$ in one of two ways:

1. If we can write b easily as a power of a then the equation becomes $a^{f(x)} = a^n$, which we can solve as shown previously.
2. We can write $f(x) = \log_a b$ and solve this logarithmic equation.

EXAMPLE 2 Solve the equations.

a. $3^{x-1} = 4$ b. $2^{x^2} = 3$ c. $3^{2x-3} = -1$

Solution

$\log_a a = 1$

$\log_a x + \log_a y = \log_a (x \cdot y)$

a. By writing the equation in logarithmic form, we get

$$3^{x-1} = 4 \Leftrightarrow x - 1 = \log_3 4 \Leftrightarrow x = \log_3 4 + 1.$$

We can keep the solution in this form or we can write it as a single logarithm:

$x = \log_3 4 + \log_3 3 \Leftrightarrow x = \log_3(4 \cdot 3) \Leftrightarrow x = \log_3 12.$

b. $2^{x^2} = 3 \Leftrightarrow x^2 = \log_2 3 \Leftrightarrow x = \pm\sqrt{\log_2 3}$

c. Since $-1 < 0$, the equation will have no solution. So the solution is the empty set.

Check Yourself 2

Solve the equations.

a. $2^{3x-2} = 3$ b. $3^{x+2} = 5$ c. $2^{-x^2} = -2$ d. $(3^x - 1)(3^x + 1) = 0$

Answers

a. $\{\log_8 12\}$ b. $\{\log_3 \frac{5}{9}\}$ c. $\varnothing$ d. $\{0\}$

3. Equations of the Form $a^{f(x)} = b^{g(x)}$

If an equation has the form $a^{f(x)} = b^{g(x)}$ and the bases a and b cannot be equalized easily, we can take the logarithms of both sides to a convenient base (usually base 10) and then solve the new equation.

EXAMPLE 3 Solve the equations.

a. $2^{x+1} = 3^{x-2}$ b. $2^{x-1} \cdot 5^{x+3} = 3$

Solution

a. Since the bases are different, we take the common logarithms of both sides:

$$\log (2^{x+1}) = \log (3^{x-2}) \Leftrightarrow (x + 1) \cdot \log 2 = (x - 2) \cdot \log 3 \Leftrightarrow$$

$$(x \cdot \log 2) + \log 2 = (x \cdot \log 3) - 2 \cdot \log 3 \Leftrightarrow \log 2 + (2 \cdot \log 3) = (x \cdot \log 3) - (x \cdot \log 2) \Leftrightarrow$$

$$\log (2 \cdot 3^2) = x \cdot \log \frac{3}{2} \Leftrightarrow x = \frac{\log 18}{\log \frac{3}{2}} \Leftrightarrow x = \log_{\frac{3}{2}} 18.$$

$\log_a b = \dfrac{\log_c b}{\log_c a}$

b. Taking the common logarithms of each side, we get

$$2^{x-1} \cdot 5^{x+3} = 3 \Leftrightarrow \log (2^{x-1} \cdot 5^{x+3}) = \log 3 \Leftrightarrow \log 2^{x-1} + \log 5^{x+3} = \log 3 \Leftrightarrow$$

$$(x-1) \cdot \log 2 + (x+3) \cdot \log 5 = \log 3 \Leftrightarrow$$

$$(x \cdot \log 2) - \log 2 + (x \cdot \log 5) + 3 \log 5 = \log 3 \Leftrightarrow$$

$$x \cdot (\log 2 + \log 5) = \log 3 + \log 2 - (3 \cdot \log 5) \Leftrightarrow x \cdot \underbrace{\log (2 \cdot 5)}_{1} = \log \frac{6}{125}, \text{ i.e.}$$

$$x = \log \frac{6}{125}.$$

Check Yourself 3

Solve the equations.

a. $3^{x+1} = 5^{2x-1}$ b. $3^{x+1} \cdot 3^{x-1} = 5$ c. $2^{2x} \cdot 3^{3x} = 5^{5x} \cdot 7^{7x}$

Answers

a. $\{\log_{\frac{25}{3}} 15\}$ b. $\{\log_9 5\}$ c. $\{0\}$

4. Working with Exponential Equations

Not all exponential equations fall into the three categories we have seen. If an equation is in a different form, we use methods such as factorization, substitution and division to obtain an equivalent equation which we can solve.

EXAMPLE 4 Solve the equations.

a. $3^{2x} - 3^x = 72$ b. $4^x - 2^{x+4} + 48 = 0$

Solution a. By rewriting the given equation as $(3^x)^2 - 3^x = 72$ and using the substitution $y = 3^x$, we get the quadratic equation

$$y^2 - y = 72 \Leftrightarrow y^2 - y - 72 = 0.$$

We can solve this by factorization:

$$(y-9) \cdot (y+8) = 0 \Leftrightarrow y_1 = 9 \text{ and } y_2 = -8.$$

$a^x > 0$

Now we must be careful. Since $y = 3^x$ is always positive, we must reject the negative solution $y_2 = -8$. Only $y = 9$ will be used to solve the original equation:

$$3^x = y = 9 \Leftrightarrow 3^x = 3^2 \Leftrightarrow x = 2.$$

So the solution set is $S = \{2\}$.

b. First we rewrite the equation:

$$4^x - 2^{x+4} + 48 = 0 \Leftrightarrow 2^{2x} - (2^4 \cdot 2^x) + 48 = 0 \Leftrightarrow (2^x)^2 - (16 \cdot 2^x) + 48 = 0.$$

Now we can substitute $y = 2^x$ and get a quadratic equation:

$$y^2 - 16y + 48 = 0 \Leftrightarrow (y-4) \cdot (y-12) = 0 \Leftrightarrow y_1 = 4 \text{ and } y_2 = 12.$$

Back substituting both $y = 4$ and $y = 12$, we get

$$2^x = y = 4 \Leftrightarrow 2^x = 2^2 \Leftrightarrow x = 2 \text{ and } 2^x = y = 12 \Leftrightarrow x = \log_2 12.$$

So the solution set is $S = \{2, \log_2 12\}$.

Check Yourself 4

Solve the equations.

a. $4^x - (9 \cdot 2^x) + 8 = 0$ b. $25^{-x} + 5^{-x+1} - 50 = 0$

Answers

a. $\{0, 3\}$ b. $\{-1\}$

EXAMPLE 5 Solve $3^{3x} + 3^{1-3x} - 4 = 0$ for x.

Solution If we substitute $3^{3x} = y$, we get $y + \dfrac{3}{y} - 4 = 0$, i.e. $y^2 - 4y + 3 = 0$.

Factorizing this as $(y - 1) \cdot (y - 3) = 0$ gives us the solutions $y_1 = 1$ and $y_2 = 3$. Since both solutions are positive, we need to solve both $3^{3x} = 1$ and $3^{3x} = 3$:

$$3^{3x} = 1 \Leftrightarrow 3x = 0 \Leftrightarrow x = 0 \text{ and } 3^{3x} = 3 \Leftrightarrow 3x = 1 \Leftrightarrow x = \frac{1}{3}.$$

So the solution set is $S = \{0, \dfrac{1}{3}\}$.

Check Yourself 5

Solve the equations.

a. $2^{x+1} - 2^{2-x} = 7$ b. $(2 + \sqrt{3})^x + (2 - \sqrt{3})^x = 14$

Answers

a. $\{2\}$ b. $\{-2, 2\}$

EXAMPLE 6 Solve the equations.

a. $2^{x+4} + 2^{x+2} = 5^{x+1} + (3 \cdot 5^x)$ b. $4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$

Solution a. $2^{x+4} + 2^{x+2} = 5^{x+1} + (3 \cdot 5^x) \Leftrightarrow (2^x \cdot 2^4) + (2^x \cdot 2^2) = (5^x \cdot 5) + (3 \cdot 5^x) \Leftrightarrow$

$$2^x \cdot (16 + 4) = 5^x \cdot (5 + 3) \Leftrightarrow \frac{2^x}{5^x} = \frac{8}{20} \Leftrightarrow (\frac{2}{5})^x = \frac{2}{5} \Leftrightarrow x = 1$$

b. We begin by rearranging the equation so that similar bases are grouped on the same side:

$$4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1} \Leftrightarrow 4^x + 2^{2x-1} = 3^{x+\frac{1}{2}} + 3^{x-\frac{1}{2}} \Leftrightarrow 4^x + \frac{2^{2x}}{2} = (3^x \cdot 3^{\frac{1}{2}}) + \frac{3^x}{3^{\frac{1}{2}}} \Leftrightarrow$$

$$4^x + \frac{4^x}{2} = 3^x\sqrt{3} + \frac{3^x}{\sqrt{3}} \Leftrightarrow 4^x \cdot (1 + \frac{1}{2}) = 3^x \cdot (\sqrt{3} + \frac{1}{\sqrt{3}}) \Leftrightarrow$$

$$\frac{3}{2} \cdot 4^x = \frac{4}{\sqrt{3}} \cdot 3^x \Leftrightarrow \frac{4^x}{3^x} = \frac{\frac{4}{\sqrt{3}}}{\frac{3}{2}} \Leftrightarrow (\frac{4}{3})^x = \frac{8}{3\sqrt{3}} = \sqrt{\frac{64}{27}} = (\frac{4}{3})^{\frac{3}{2}} \Leftrightarrow x = \frac{3}{2}.$$

Check Yourself 6

Solve the equations.

a. $3^{x+1} \cdot 5^x = 2^{2x} \cdot 7^{x+2}$ b. $5^x + 5^{x+1} + 5^{x+2} = 3^x + 3^{x+1} + 3^{x+2}$

Answers

a. $\{\log_{\frac{15}{28}} \frac{49}{3}\}$ b. $\{\log_{\frac{5}{3}} \frac{13}{31}\}$

EXAMPLE 7 Solve the equations.

a. $9^x + 4^x = \frac{5}{2} \cdot 6^x$ b. $(10 \cdot 25^{\frac{1}{x}}) + (10 \cdot 4^{\frac{1}{x}}) - (29 \cdot 10^{\frac{1}{x}}) = 0$

Solution a. $9^x + 4^x = \frac{5}{2} \cdot 6^x \Leftrightarrow 3^{2x} + 2^{2x} - \frac{5}{2} \cdot 2^x \cdot 3^x = 0$

Let us divide both sides by 2^{2x} (note that $2^{2x} \neq 0$):

$$\frac{3^{2x}}{2^{2x}} + \frac{2^{2x}}{2^{2x}} - \left[\frac{5}{2} \cdot \frac{2^x \cdot 3^x}{2^{2x}}\right] = \frac{0}{2^{2x}} \Leftrightarrow (\frac{3}{2})^{2x} + 1 - \left[\frac{5}{2} \cdot (\frac{3}{2})^x\right] = 0.$$

Substituting $y = (\frac{3}{2})^x$, we obtain $y^2 - \frac{5}{2}y + 1 = 0$ or $2y^2 - 5y + 2 = 0$, which we can factorize as $(2y - 1)(y - 2) = 0$.

This equation has solutions $y_1 = \frac{1}{2}$ and $y_2 = 2$, which are both positive. So we solve $y = (\frac{3}{2})^x$ considering both cases:

$$(\frac{3}{2})^x = \frac{1}{2} \Leftrightarrow x = \log_{\frac{3}{2}} \frac{1}{2} \text{ (i.e. } x = -\log_{\frac{3}{2}} 2) \text{ and } (\frac{3}{2})^x = 2 \Leftrightarrow x = \log_{\frac{3}{2}} 2.$$

In conclusion, the solution set is $S = \{\pm \log_{\frac{3}{2}} 2\}$.

b. Following similar steps, we get

$$(10 \cdot 25^{\frac{1}{x}}) + (10 \cdot 4^{\frac{1}{x}}) - (29 \cdot 10^{\frac{1}{x}}) = 0 \Leftrightarrow (10 \cdot 5^{2 \cdot \frac{1}{x}}) + (10 \cdot 2^{2 \cdot \frac{1}{x}}) - \left[29 \cdot (5 \cdot 2)^{\frac{1}{x}}\right] = 0.$$

Dividing both sides by $(5 \cdot 2)^{\frac{1}{x}} = 5^{\frac{1}{x}} \cdot 2^{\frac{1}{x}}$ gives us

$$\left[10 \cdot (\frac{5}{2})^{\frac{1}{x}}\right] + \left[10 \cdot (\frac{2}{5})^{\frac{1}{x}}\right] - 29 = 0 \Leftrightarrow \left[10 \cdot (\frac{5}{2})^{\frac{1}{x}}\right] + \left[10 \cdot \frac{1}{(\frac{5}{2})^{\frac{1}{x}}}\right] - 29 = 0.$$

Substituting $(\frac{5}{2})^{\frac{1}{x}} = y$, we obtain

$$10y + \frac{10}{y} - 29 = 0 \Leftrightarrow 10y^2 - 29y + 10 = 0 \Leftrightarrow (5y - 2)(2y - 5) = 0.$$

This has solutions $y_1 = \frac{2}{5}$ and $y_2 = \frac{5}{2}$. Therefore, we solve the equations

$$(\frac{5}{2})^{\frac{1}{x}} = \frac{2}{5} \Leftrightarrow (\frac{5}{2})^{\frac{1}{x}} = (\frac{5}{2})^{-1} \Leftrightarrow \frac{1}{x} = -1 \Leftrightarrow x = -1 \text{ and } (\frac{5}{2})^{\frac{1}{x}} = \frac{5}{2} \Leftrightarrow \frac{1}{x} = 1 \Leftrightarrow x = 1.$$

Hence the solution set is $S = \{\pm 1\}$.

Note

We can generalize the approach we used in the previous example as follows: for an exponential equation of the form $(p \cdot a^{2f(x)}) + (q \cdot b^{2f(x)}) + (r \cdot a^{f(x)} \cdot b^{f(x)}) = 0$, we divide both sides by any of its terms (excluding the coefficient) in order to obtain a known type of equation (quadratic, cubic, etc).

For example, dividing all terms by $b^{2f(x)}$ gives us $p \cdot (\frac{a}{b})^{2f(x)} + r \cdot (\frac{a}{b})^{f(x)} + q = 0$, which becomes a quadratic equation $(p \cdot y^2) + (r \cdot y) + q = 0$ with the substitution $y = (\frac{a}{b})^{f(x)}$. By considering only the positive solutions to this equation $(y = (\frac{a}{b})^{f(x)} > 0)$, we can solve the original exponential equation using $(\frac{a}{b})^{f(x)} = y$.

Check Yourself 7

Solve the equations.

a. $(3 \cdot 16^x) + (37 \cdot 36^x) = 26 \cdot 81^x$ b. $(4 \cdot 3^x) - (9 \cdot 2^x) = 5 \cdot 6^{\frac{x}{2}}$

Answers

a. $\{\frac{1}{2}\}$ b. $\{4\}$

EXAMPLE 8 Solve $2^{3x+1} + 1 = 4^x + 2^{x+1}$ for x.

Solution $2^{3x+1} + 1 = 4^x + 2^{x+1} \Leftrightarrow (2^{x+1} \cdot 2^{2x}) + 1 - 2^{2x} - 2^{x+1} = 0 \Leftrightarrow$
$(2^{2x} \cdot 2^{x+1} - 2^{2x}) - (2^{x+1} - 1) = 0 \Leftrightarrow [2^{2x} \cdot (2^{x+1} - 1)] - (2^{x+1} - 1) = 0 \Leftrightarrow$
$(2^{x+1} - 1) \cdot (2^{2x} - 1) = 0.$

Solving for each factor, we get

$2^{x+1} - 1 = 0 \Leftrightarrow 2^{x+1} = 1 \Leftrightarrow x + 1 = 0 \Leftrightarrow x = -1$ or

$2^{2x} - 1 = 0 \Leftrightarrow 2^{2x} = 1 \Leftrightarrow 2x = 0 \Leftrightarrow x = 0.$

So the solution set is $S = \{-1, 0\}$.

Check Yourself 8

Solve the equations.

a. $5^{1+2x} + 6^{1+x} - 150^x = 30$ b. $|9 \cdot (9^x + 9^{-x})| - |3 \cdot (3^x + 3^{-x})| = 72$

Answers

a. $\{\log_6 5, \log_{25} 6\}$ b. $\{-1, 1\}$

5. Exponential Equations with a Unique Solution

When we are asked to solve an exponential equation of the form $f(x) = c$ in which $f(x)$ is strictly monotone (either increasing or decreasing), note that the function will have the value c at a unique value of x. Therefore, if we can identify a value of x which satisfies the given equation using any method (even by guessing), we can conclude that it is the only solution to the equation.

We can apply a similar logic to equations of the form $f(x) = g(x)$ where f and g have different behaviors. In other words, if one function is strictly increasing and the other is strictly decreasing then we can say that this equation has a unique solution.

EXAMPLE 9 Solve the equations.

a. $3^x + 4^x = 7$ b. $3^x + 4^x = 5^x$

Solution a. Since an exponential function with base greater than one is strictly increasing and the sum of two strictly increasing functions is also strictly increasing, we can say that $3^x + 4^x$ is a strictly increasing function. In addition, the right-hand side of the equation is a constant number. Therefore the equation has the form $f(x) = c$ for a monotone function $f(x)$ and there is a unique solution for the given equation. In addition, notice that for $x = 1$, $3^x + 4^x = 3^1 + 4^1 = 7$. So $x = 1$ is the unique solution for this equation.

b. Both sides of the equation are strictly increasing functions, so we cannot come to any direct conclusion. However, if we divide both sides by 5^x, we obtain $\frac{3^x}{5^x}+\frac{4^x}{5^x}=1$ or $(\frac{3}{5})^x+(\frac{4}{5})^x=1$.

Since the left-hand side is the sum of two exponential functions which are strictly decreasing (because their bases are less than 1), it is strictly decreasing. In addition, the right-hand side is constant. Therefore the equation has a unique solution which we can identify as $x = 2$ (since $(\frac{3}{5})^2+(\frac{4}{5})^2=\frac{9}{25}+\frac{16}{25}=\frac{25}{25}=1$).

Check Yourself 9

Solve the equations.

a. $3^x + 4^x = 91$ b. $3^{x-1} + 5^{x-1} = 34$

Answers

a. $\{3\}$ b. $\{3\}$

6. Equations of the Form $f(x)^{g(x)} = f(x)^{h(x)}$

When solving equations of the form $f(x)^{g(x)} = f(x)^{h(x)}$, we consider the following cases:

1. $f(x) = 1$ When the base of an exponential expression is 1 then no matter what value the exponent has, the expression will equal 1. So we just need to solve $f(x) = 1$ and include this solution in the solution set.

2. $f(x) = -1$ When the base is -1, we need to check whether the resulting value(s) of x satisfy $(-1)^{g(x)} = (-1)^{h(x)}$ or not. In other words, after solving $f(x) = -1$, we check the values of $g(x)$ and $h(x)$ for the obtained solutions. We include the values of x which satisfy $(-1)^{g(x)} = (-1)^{h(x)}$ in the solution set.

3. $f(x) = 0$ For each value of x which makes the base zero, we need to check the sign (positive or negative) of the exponents. We include any values of x which satisfy $f(x) = 0$, $g(x) > 0$ and $h(x) > 0$ in the solution set.

4. $g(x) = h(x)$ We need to include the solutions of the equation $g(x) = h(x)$ in the solution set.

EXAMPLE 10 Solve the equations.

a. $x^{x+4} = x$ b. $(3x-4)^{2x^2+2} = (3x-4)^{5x}$

Solution a. We consider the four cases above:

1. $x = 1$ We include this as a solution in the solution set.
2. $x = -1$ We need to check $(-1)^{-1+4} = (-1)$. Since this is true, -1 will be in the solution set.
3. $x = 0$ Since the exponents are $x + 4 = 0 + 4 = 4 > 0$ and $1 > 0$, we include zero the solution set.
4. $x + 4 = 1$ This gives us the additional solution $x = -3$, which we include in the solution set.

In conclusion, the solution set is $S = \{-3, -1, 0, 1\}$.

b. We check the four cases:

1. $3x - 4 = 1$ This has the solution $x = \frac{5}{3}$, which we add to the solution set.
2. $3x - 4 = -1$ Solving this linear equation gives us $x = 1$. So we check the exponents for this value: $2x^2 + 2 = 2 \cdot (1)^2 + 2 = 4$ and $5x = 5 \cdot 1 = 5$.

   Since $(-1)^4 \neq (-1)^5$, we do not include $x = 1$ as a solution.
3. $3x - 4 = 0$ We need to confirm that the exponents are positive for $x = \frac{4}{3}$ in order to include this value as a solution:

   $$2x^2 + 2 = 2 \cdot (\frac{4}{3})^2 + 2 = \frac{50}{9} > 0 \text{ and } 5 \cdot x = 5 \cdot \frac{4}{3} = \frac{20}{3} > 0.$$

   So $x = \frac{4}{3}$ is in the solution set.
4. $2x^2 + 2 = 5x$ Solving this quadratic equation gives us

   $2x^2 - 5x + 2 = 0 \Leftrightarrow (2x-1) \cdot (x-2) = 0 \Leftrightarrow x = \frac{1}{2}$ and $x = 2$.

In conclusion, the solution set for the equation $(3x-4)^{2x^2+2} = (3x-4)^{5x}$ is $S = \{\frac{1}{2}, \frac{4}{3}, \frac{5}{3}, 2\}$.

Check Yourself 10

Solve the equations.

a. $(x+1)^x = 1$ b. $(x-3)^{\frac{x+1}{4}} = (x-3)^{\frac{x-2}{3}}$

Answers

a. $\{0, -2\}$ b. $\{3, 4, 11\}$

7. Exponential Equations with Parameters

In this section we will look at how to solve exponential equations which contain parameters.

EXAMPLE 11 Determine all values of a for which $2^x = \dfrac{2a+5}{a-2}$ has no solution.

Solution Since the left-hand side of this equation is always positive, the equation will have no solution for any values of a which make the right-hand right side negative or zero. In other words, we need to solve $\dfrac{2a+5}{a-2} \leq 0$.

By making a sign table or by any other method, we decide that the values $a \in [-\frac{5}{2}, 2)$ make $\dfrac{2a+5}{a-2}$ zero or negative,

x	$-\infty$		$-\frac{5}{2}$		2		∞
$\frac{2a+5}{a-2}$		+	•	−		+	

and therefore $2^x = \dfrac{2a+5}{a-2}$ has no solution for these values.

EXAMPLE 12 Determine all real values of the parameter m for which the equation $4^x - (m \cdot 2^x) - m + 3 = 0$ has only one solution.

Solution Substituting $2^x = y$ gives us $y^2 - my - m + 3 = 0$.

Since exponential functions are one-to-one, the original equation will have a unique solution if and only if this quadratic equation has only one positive solution for y.

There are two cases in which there is only one positive solution for y: either the quadratic equation has two roots with opposite signs, or it has a positive double root. We can express this mathematically by writing

$$\text{either } \begin{cases} \Delta > 0 \\ P < 0 \end{cases} \quad (1) \qquad \text{or} \qquad \begin{cases} \Delta = 0 \\ S > 0 \end{cases} \quad (2)$$

For $ax^2 + bx + c = 0$

$\Delta = b^2 - 4ac$

$P = \dfrac{c}{a}$

$S = -\dfrac{b}{a}$.

where Δ is the discriminant, P is the product and S is the sum of the roots of the quadratic equation. In other words,

$$\Delta = (-m)^2 - 4 \cdot 1 \cdot (-m+3) = m^2 + 4m - 12 = (m+6) \cdot (m-2),$$

$$P = -m + 3 \text{ and } S = m.$$

Replacing these values in (1) gives us

$$\begin{cases}(m+6)\cdot(m-2)>0\\ -m+3<0,\end{cases}$$

from which we conclude $m > 3$. Similarly, (2) becomes

$$\begin{cases}(m+6)\cdot(m-2)=0\\ m>0,\end{cases}$$

which gives us $m = 2$. Therefore, $4^x - (m \cdot 2^x) - m + 3 = 0$ has only one solution if $m \in \{2\} \cup (3, \infty)$.

Check Yourself 11

1. Determine all values of a for which the equation $9^x + a \cdot (a + 1) = (2a + 1) \cdot 3^x$ has

   a. no solution. b. only one solution. c. two solutions.

2. State the solutions of the equation $4^x - 2m(m + 1) \cdot 2^{x-1} + m^3 = 0$ in terms of the parameter m.

Answers

1. a. $(-\infty, -1]$ b. $(-1, 0]$ c. $(0, \infty)$

2. $S = \begin{cases}\{\log_2 m^2\} & \text{for } m \in (-\infty, 0)\\ \varnothing & \text{for } m = 0\\ \{\log_2 m, \log_2 m^2\} & \text{for } m \in (0, \infty)\end{cases}$

B. EXPONENTIAL INEQUALITIES

To solve simple exponential inequalities, we use the monotone property of exponential functions which we saw earlier in this module:

$$\text{for } a > 1,\ a^{g(x)} > a^{h(x)} \Leftrightarrow g(x) > h(x), \text{ and}$$

$$\text{for } 0 < a < 1,\ a^{g(x)} > a^{h(x)} \Leftrightarrow g(x) < h(x).$$

Notice that for $a > 1$ the directions of the inequalities are the same, but for $0 < a < 1$ they are opposite.

EXAMPLE 13 Solve the inequalities.

a. $2^{4x} < 16$ b. $3^x \cdot (\frac{1}{3})^{x-3} \leq (\frac{1}{27})^x$

Solution a. We can write $2^{4x} < 16 \Leftrightarrow 2^{4x} < 2^4$. Since the base (2) is greater than 1, the direction of inequality for the exponents will be the same: $4x < 4 \Leftrightarrow x < 1$.

So the inequality holds for $x \in (-\infty, 1)$.

b. Rewriting the given inequality gives us

$$3^x \cdot (\frac{1}{3})^{x-3} \le (\frac{1}{27})^x \Leftrightarrow 3^x \cdot 3^{3-x} \le 3^{-3x} \Leftrightarrow 3^{x+3-x} \le 3^{-3x} \Leftrightarrow 3^3 \le 3^{-3x}.$$

Since the base is greater than 1, we can write $3 \le -3x$ which means $-1 \ge x$. Hence, $x \in (-\infty, -1]$.

Check Yourself 12

Solve the inequalities.

a. $2^{4x} < 32$ b. $(\frac{2}{3})^{2-2x} \le (\frac{8}{27})^{x-2}$

Answers

a. $(-\infty, \frac{5}{4})$ b. $(-\infty, \frac{8}{5}]$

The examples above solve simple inequalities. To solve more complex inequalities, follow the steps:

A critical value of $f(x)$ is a value of x which makes the expression $f(x) = 0$ true.

1. Rewrite the inequality so that one side becomes zero: $f(x) \ge 0$ or $f(x) \le 0$.
2. Solve $f(x) = 0$ for critical values of $f(x)$.
3. Establish a sign table for $f(x)$ with these critical values. To determine the sign for each interval, we place the sign of the leading coefficient in the rightmost interval and change the sign at each single root while moving left through the intervals. Alternatively, we can use a value of x from each interval to check the sign of the function in that interval.
4. Select the appropriate intervals as the solution set.

Let us look at some examples.

EXAMPLE 14 Solve the inequalities.

a. $4^{\frac{5x-1}{2x-1}} \ge 64$ b. $9^x - 5 \cdot 3^x + 6 \le 0$

Solution a. We follow the steps described above:

1. Considering that $64 = 4^3$ and $4 > 1$, we get

$$4^{\frac{5x-1}{2x-1}} \ge 4^3 \Leftrightarrow \frac{5x-1}{2x-1} \ge 3 \Leftrightarrow \frac{5x-1}{2x-1} - 3 \ge 0 \Leftrightarrow \frac{5x-1-6x+3}{2x-1} \ge 0 \Leftrightarrow \frac{-x+2}{2x-1} \ge 0.$$

2. Find the critical values: $-x + 2 = 0 \Leftrightarrow x = 2$ and $2x - 1 = 0 \Leftrightarrow x = \frac{1}{2}$.
3. Since $-x + 2$ has a negative leading coefficient and $2x - 1$ has a positive leading coefficient, the fraction in which they are numerator and denominator will have a negative leading coefficient. Therefore, when we draw the sign table, we start with a negative sign in the rightmost interval and alternate as we move left.

x	$-\infty$	$\frac{1}{2}$		2		∞
$\frac{-x+2}{2x-1}$	$-$		$+$	$\bullet$	$-$	

4. We select the interval $(\frac{1}{2}, 2]$ as the solution set for x, because it is the solution set of $\frac{-x+2}{2x-1} \geq 0$, which implies that the fraction should be positive.

b. Let $f(x) = 9^x - 5 \cdot 3^x + 6$. Factorizing $f(x)$ gives us

$$f(x) = (3^x)^2 - 5 \cdot 3^x + 6 \Leftrightarrow f(x) = (3^x - 3) \cdot (3^x - 2).$$

$\log_3 2 < \log_3 3$
$\log_3 2 < 1$

If we solve $f(x) = 0$, we get

$$3^x - 3 = 0 \Leftrightarrow 3^x = 3 \Leftrightarrow x_1 = 1 \text{ and } 3^x - 2 = 0 \Leftrightarrow 3^x = 2 \Leftrightarrow x_2 = \log_3 2.$$

Let us establish a sign table with these values:

x	$-\infty$	$\log_3 2$		1		∞
$f(x)$	$+$	$\bullet$	$-$	$\bullet$	$+$	

We can see that $f(x) \leq 0$ if $x \in [\log_3 2, 1]$.

Check Yourself 13

Solve the inequalities.

a. $25^x < (6 \cdot 5^x) - 5$ b. $4^x - (2 \cdot 25^x) < 10^x$

Answers

a. $(0, 1)$ b. $(\log_{\frac{2}{5}} 2, \infty)$

EXAMPLE 15 Find the value(s) of x which satisfy $|x|^{x^2-3x-4} < 1$.

Solution By definition, $|x| \geq 0$ and we can exclude the case $|x| = 0$ as this would result in a negative exponent and therefore a division by zero (since $0^{-4} = \frac{1}{0^{-4}} = \frac{1}{0}$). So we can write $|x|^{x^2-3x-4} < 1 \Leftrightarrow |x|^{x^2-3x-4} < |x|^0$.

Now we have to check the cases $0 < |x| < 1$ and $|x| > 1$ (for $|x| = 1$, the inequality will be $1 < 1$, which is false). In other words, we have to solve the systems

$a^{f(x)} < a^{g(x)} \Leftrightarrow$
$f(x) < g(x)$ if $a > 1$
$f(x) > g(x)$ if $0 < a < 1$

$$\begin{cases} 0 < |x| < 1 \\ x^2 - 3x - 4 > 0 \end{cases} \Leftrightarrow \begin{cases} -1 < x < 1 \text{ and } x \neq 0 \\ x < -1 \text{ or } x > 4 \end{cases} \Rightarrow \text{no solution}$$

and $$\begin{cases} |x| > 1 \\ x^2 - 3x - 4 < 0 \end{cases} \Leftrightarrow \begin{cases} x < -1 \text{ or } x > 1 \\ -1 < x < 4 \end{cases} \Rightarrow x \in (1, 4).$$ So the solution set is $x \in (1, 4)$.

EXAMPLE 16 Solve the inequality $a^2 - 9^{x+1} - (8a \cdot 3^x) > 0$ for negative, zero and positive real values of a.

Solution We begin by factorizing the left-hand side:

$$a^2 - 9^{x+1} - (8a \cdot 3^x) > 0 \Leftrightarrow (9 \cdot 9^x) + (8a \cdot 3^x) - a^2 < 0 \Leftrightarrow$$
$$9 \cdot (3^x)^2 + (8a \cdot 3^x) - a^2 < 0 \Leftrightarrow (3^{x+2} - a) \cdot (3^x + a) < 0.$$

Let us look at the solution to this inequality for each of the three different cases.

- $a > 0$: If we try to find the critical values of the expression on the left-hand side, we get
$$(3^{x+2} - a) \cdot (3^x + a) = 0 \;\Leftrightarrow\; (3^{x+2}) - a = 0 \;\Leftrightarrow\; x = \log_3 \frac{a}{9} = -2 + \log_3 a.$$
  Notice that $3^x + a > 0$. The corresponding sign table is therefore

x	$-\infty$		$-2 + \log_3 a$		∞
$(3^{x+2} - a) \cdot (3^x + a)$		$-$	$\circ$	$+$	

  As a result, $(3^{x+2} - a) \cdot (3^x + a) < 0$ if $x \in (-\infty, -2 + \log_3 a)$ for $a > 0$.

- $a = 0$: Since both 3^{x+2} and 3^x are positive, their product will be positive. So the inequality cannot be negative and therefore it has no solution when $a = 0$.

- $a < 0$: Since $3^{x+2} - a > 0$, we conclude that $3^x + a$ should be zero while finding the critical value(s) of the product $(3^{x+2} - a) \cdot (3^x + a)$. So we get the solution $x = \log_3(-a)$ for $(3^{x+2} - a) \cdot (3^x + a) = 0$. The corresponding sign table is

x	$-\infty$		$\log_3(-a)$		∞
$(3^{x+2} - a) \cdot (3^x + a)$		$-$	$\circ$	$+$	

  So, $(3^{x+2} - a) \cdot (3^x + a) < 0$ if $x \in (-\infty, \log_3(-a))$ for $a < 0$.

In conclusion, for $a^2 - 9^{x+1} - 8a \cdot 3^x > 0$, $S = \begin{cases} x \in (-\infty, \log_3(-a)) & \text{when } a < 0 \\ \varnothing & \text{when } a = 0 \\ x \in (-\infty, -2 + \log_3 a) & \text{when } a > 0. \end{cases}$

Check Yourself 14

1. Solve $(x^2 + x + 1)^x < 1$.
2. Solve $4^x - a \cdot 2^x + 3 - a \leq 0$ for different real values of the parameter a.

Answers

1. $(-\infty, -1)$

2. $S = \begin{cases} \varnothing & \text{for } a \in (-\infty, 2) \\ 0 & \text{for } a = 2 \\ \left[\log_2 \dfrac{a - \sqrt{a^2 + 4a - 12}}{2}, \log_2 \dfrac{a + \sqrt{a^2 + 4a - 12}}{2}\right] & \text{for } a \in (2, 3) \\ \left(-\infty, \log_2 \dfrac{a + \sqrt{a^2 + 4a - 12}}{2}\right] & \text{for } a \in [3, \infty) \end{cases}$

EXERCISES 3.1

A. Exponential Equations

In questions 1-9, solve the equations for x.

1. a. $4^x = 8$ b. $2^{3x-1} = \frac{1}{64}$

c. $5^{x^2-x} = 25$ d. $3^{2x-1} = \sqrt{3^x}$

e. $2^{3x} \cdot 5^x = 1600$ f. $6^{2x+4} = 3^{3x} \cdot 2^{x+8}$

g. $7^{\frac{x}{4}+\frac{1}{x}-1} = 1$ h. $27^{\frac{2x-1}{x}} = 9^{\frac{2x-1}{2}}$

2. a. $2^{x-1} = 5$ b. $3^{x^2} = 1$

c. $5^{x^2} = 4$ d. $3^{2x-1} = 2$

e. $2^{3x+2} = 3$ f. $3^{2x} - 4 = 0$

3. a. $2^x \cdot 3^{x+1} 5^{x-2} = 7$ b. $2^{4x} \cdot 2^{2x} \cdot 5^x = 1$

c. $5^{3x+1} = 8^{2x-1}$ d. $3^x \cdot 5^x = 7^{2x}$

e. $5^{3(x+2)} = 9^{x+2}$

4. a. $5^{2x} - (6 \cdot 5^x) = 475$

b. $3^{2x+5} = 3^{x+2} + 2$

c. $4^{2x-2} - (6 \cdot 4^{x-2}) - 1 = 0$

d. $9^x - 3^{x+2} + 14 = 0$

✪e. $(1 + \sqrt{3})^x + 2^{x-1} \cdot (2 + \sqrt{3})^x = 4$

5. a. $\frac{e^x}{2} - \frac{4}{e^x} = 1$

b. $3^{x+1} + 3^x = 324$

c. $(2 \cdot 5^{x+2}) + 5^{x-1} = 251$

d. $3^x \cdot 5^x = 2^{2x+1} \cdot 7^{x+2}$

6. a. $(4 \cdot 2^{2x}) - 6^x - (18 \cdot 3^{2x}) = 0$

b. $(6 \cdot 9^{\frac{1}{x}}) + (6 \cdot 4^{\frac{1}{x}}) = 13 \cdot 6^{\frac{1}{x}}$

c. $(7 \cdot 4^{x^2}) + (2 \cdot 49^{x^2}) = 9 \cdot 14^{x^2}$

d. $25^x - (3 \cdot 10^x) + (2 \cdot 4^x) = 0$

✪e. $(5 \cdot 50^x) + 8^x = (2 \cdot 125^x) + (4 \cdot 20^x)$

✪**7.** a. $2^x + 1 = (\frac{1}{2})^{2x} + (\frac{1}{2})^x$

b. $2^x - 3^x = \sqrt{6^x - 9^x}$

c. $3^{2x} + \frac{36}{3^{2x}} - (3^x + \frac{6}{3^x}) - 8 = 0$

8. a. $2^x + 3^x = 35$ b. $3 + 5^{x/3} = 2^x$

9. a. $x^{\sqrt{x}} = \sqrt{x^x}$ b. $x^{x^2-3x+2} = 1$

B. Exponential Inequalities

In questions 10–13, solve the inequalities.

10. a. $5^x > 3125$ b. $2^{3x-1} = (\frac{1}{64})^x > \sqrt{\frac{1}{8}}$

c. $5^{x-3} \geq 7^{3-x}$ d. $15^{2x+4} - 3^{3x} \cdot 5^{4x-4} \leq 0$

e. $3^x + 2^{x-1} + 2^{x+2} - 3^{x-1} + 2^{x-3} \geq 0$

11. a. $\frac{3^x}{3^x - 2^x} > 3$

b. $2^x + 3^x + 4^x \geq 29$

c. $40^x - 9^x < 16^x + 15^x$

12. a. $0.1^{x+1} < 0.8 + 2 \cdot 10^x$

b. $25^{-x} + 5^{-x+1} \geq 50$

c. $(3 \cdot 16^x) + (2 \cdot 81^x) - (5 \cdot 36^x) < 0$

13. a. $\frac{6}{2^x - 1} < 2^x$ b. $\frac{120}{1 - 25^{-x}} \leq \frac{1}{5^{-x-2}}$

14. $(4x^2 + 2x + 1)^{x^2-x} < 1$

2 LOGARITHMIC EQUATIONS AND INEQUALITIES

A. LOGARITHMIC EQUATIONS

An equation containing logarithms which have the variable in their base or argument (or both) is called a logarithmic equation. Here are some examples of logarithmic equations:

$$\log_2 x = 3,\ \log_2(x \cdot \sqrt{x+2}) = 2,\ \log_x 3 + \log_x 9 = -3,\ \log_x(2x + 3) = 2.$$

When solving logarithmic equations, we can use the following properties of logarithms:

1. $\log_a(f(x) \cdot g(x)) = \log_a|f(x)| + \log_a|g(x)|$
2. $\log_a(\dfrac{f(x)}{g(x)}) = \log_a|f(x)| - \log_a|g(x)|$
3. $\log_a f^n(x) = n \cdot \log_a|f(x)|$.

We must also remember that the argument of a logarithm is always positive, and its base is always positive and different from 1. As we have seen, these conditions are called the existence conditions for logarithms.

Just as we saw for exponential equations, there is no single method that we can use to solve all logarithmic equations. Instead, we can consider some common types of equation and their solution. As a general rule, after solving a given equation we always need to check which solutions satisfy the existence conditions in order to complete the solution.

1. Equations of the Form $\log_{f(x)} g(x) = b$

To solve an equation of this type, we can write it in exponential form using

$\log_{f(x)} g(x) = b \Leftrightarrow f(x)^b = g(x)$ and solve this new equation for x.

After finding the solution(s), we eliminate any values of x which do not satisfy the restrictions $f(x) > 0$, $f(x) \neq 1$, and $g(x) > 0$ (by the existence conditions).

Let us look at some examples.

EXAMPLE 17 Solve the equations.

a. $\log_2(3x - 1) = 3$ b. $\log_{x+1} 16 = 2$ c. $\log_3(5x^2 - 9x + 7) = 2$

Solution a. $\log_2(3x - 1) = 3 \Leftrightarrow 3x - 1 = 2^3 \Leftrightarrow 3x = 9 \Leftrightarrow \boxed{x = 3}$

By the existence conditions, since $x = 3$ satisfies $3x - 1 > 0$ it is the solution of the given logarithmic equation.

b. $\log_{x+1} 16 = 2 \Leftrightarrow (x+1)^2 = 16 \Leftrightarrow x + 1 = \pm 4 \Leftrightarrow x_1 = 3$ or $x_2 = -5$

Check against the existence conditions: we need $x + 1 > 0$ and $x + 1 \neq 1$, so we accept only $x = 3$ as the solution since $x = -5$ does not satisfy $x + 1 > 0$.

c. $\log_3(5x^2 - 9x + 7) = 2 \Leftrightarrow 5x^2 - 9x + 7 = 3^2 \Leftrightarrow 5x^2 - 9x - 2 = 0 \Leftrightarrow (5x + 1) \cdot (x - 2) = 0$

Accordingly, we get $x_1 = -\frac{1}{5}$ and $x_2 = 2$ as possible solutions. Checking for $5x^2 - 9x + 7 > 0$, we decide that both solutions satisfy the inequality. So the solution set is $S = \{-\frac{1}{5}, 2\}$.

EXAMPLE 18 Solve the equations.

a. $\log_{2x}(3x^2 - 3x + 4) = 2$ b. $\log_{\frac{1}{5}}(\log_5 \sqrt{5x}) = 0$ c. $\log_2(3 - x) + \log_2(1 - x) = 3$

Solution a. $\log_{2x}(3x^2 - 3x + 4) = 2 \Leftrightarrow (2x)^2 = 3x^2 - 3x + 4 \Leftrightarrow x^2 + 3x - 4 = 0 \Leftrightarrow$

$(x + 4) \cdot (x - 1) = 0 \Leftrightarrow x_1 = -4$ and $x_2 = 1$

If we check these solutions for the conditions $2x > 0$, $2x \neq 1$ and $3x^2 - 3x + 4 > 0$, we find that $x_1 = -4$ does not satisfy $2x > 0$. Therefore we eliminate it, and so the only solution is $x = 1$.

b. $\log_{\frac{1}{5}}(\log_5 \sqrt{5x}) = 0 \Leftrightarrow \log_5 \sqrt{5x} = (\frac{1}{5})^0 \Leftrightarrow \log_5 \sqrt{5x} = 1 \Leftrightarrow \sqrt{5x} = 5^1 \Leftrightarrow x = 5$

The existence conditions state that we must have

$5x > 0$ and $\log_5 \sqrt{5x} > 0$.

When $x = 5$, $5x = 25 > 0$ and $\log_5 \sqrt{5x} = \log_5 \sqrt{5 \cdot 5} = \log_5 5 = 1 > 0$ and so we conclude that $x = 5$ is the solution of the equation.

c. $\log_2(3 - x) + \log_2(1 - x) = 3 \Leftrightarrow \log_2|(3 - x) \cdot (1 - x)| = 3 \Leftrightarrow x^2 - 4x + 3 = 2^3 \Leftrightarrow$

$x^2 - 4x - 5 = 0 \Leftrightarrow (x - 5) \cdot (x + 1) = 0 \Leftrightarrow x_1 = -1$ and $x_2 = 5$

Checking for the conditions $3 - x > 0$ and $1 - x > 0$, we conclude that $x = 5$ does not satisfy either condition. Therefore we eliminate it. Since $x = -1$ satisfies both conditions, it is the only solution to the given equation.

Check Yourself 15

Solve the equations.

a. $\log_3(5x + 2) = 3$ b. $\log_{x-1} 49 = 2$ c. $\log_{2x^2+1} 81 = 4$

d. $\log_{x+1}(x^2 - 3x + 1) = 1$ e. $\log_{\frac{1}{3}}(\log_{\frac{1}{2}}(x+1)) = -1$

Answers

a. $\{5\}$ b. $\{8\}$ c. $\{\pm 1\}$ d. $\{4\}$ e. $\{-\frac{7}{8}\}$

2. Equations of the Form $\log_{f(x)} g(x) = \log_{f(x)} h(x)$

Since logarithmic functions are one-to-one, we can write

$$\log_{f(x)} g(x) = \log_{f(x)} h(x) \Leftrightarrow g(x) = h(x).$$

Therefore, to solve an equation of the form $\log_{f(x)} g(x) = \log_{f(x)} h(x)$, we solve the equation $g(x) = h(x)$ for x and then check the solutions for the restrictions $f(x) > 0$, $f(x) \neq 1$, $g(x) > 0$ and $h(x) > 0$.

EXAMPLE 19 Solve the equations.

a. $\log_{\frac{1}{5}} \dfrac{x+2}{10} = \log_{\frac{1}{5}} \dfrac{2}{x+1}$

b. $\log\sqrt{x+21} + \log\sqrt{x-21} = 1 + \log 2$

Solution a. $\log_{\frac{1}{5}} \dfrac{x+2}{10} = \log_{\frac{1}{5}} \dfrac{2}{x+1} \Leftrightarrow \dfrac{x+2}{10} = \dfrac{2}{x+1} \Leftrightarrow x^2 + 3x - 18 = 0 \Leftrightarrow$

$$(x+6)(x-3) = 0 \Leftrightarrow x_1 = -6 \text{ and } x_2 = 3$$

These solutions must satisfy $\dfrac{x+2}{10} > 0$, $\dfrac{2}{x+1} > 0$ and $x + 1 \neq 0$.

After checking, we eliminate $x = -6$ and take $x = 3$ as the solution.

b. Simplifying the given equation gives us

$$\log\sqrt{x+21} + \log\sqrt{x-21} = 1 + \log 2 \Leftrightarrow \log(\sqrt{x+21} \cdot \sqrt{x-21}) = \log 10 + \log 2 \Leftrightarrow$$

$$\log\sqrt{x^2 - 21^2} = \log(10 \cdot 2) \Leftrightarrow \sqrt{x^2 - 441} = 20 \Leftrightarrow x^2 = 841 \Leftrightarrow x = \pm 29.$$

We eliminate $x = -29$ because it does not satisfy the condition $\sqrt{x \pm 21} > 0$.

Therefore, $x = 29$ is the only solution for the given equation.

Check Yourself 16

Solve the equations.

a. $\log_3(x^2 - 4x + 3) = \log_3(3x + 21)$

b. $\log_5(x-2) + \log_5 x = \log_5 8$

c. $\log_2 \dfrac{x-2}{x-1} = 1 + \log_2 \dfrac{3x-7}{3x-1}$

Answers

a. $\{-2, 9\}$ b. $\{4\}$ c. $\{3\}$

3. Working with Logarithmic Equations

If a logarithmic equation is not already in one of the two forms $\log_{f(x)} g(x) = b$ or $\log_{f(x)} g(x) = \log_{f(x)} h(x)$, we can sometimes rewrite it in one of these forms by using substitution and the laws of exponents.

Let us look at some examples.

EXAMPLE 20 Solve the equations.

a. $\log_3^2 x + 2\log_3 x = 3$ b. $x^{\log_3 x} = 3$ c. $\log_2(12 - 2^x) = 5 - x$

Solution a. Substituting $\log_3 x = y$ gives us a quadratic equation:

$\log_a^2 x = \log_a x \cdot \log_a x$
$\log_a^n x = (\log_a x)^n$

$$\log_3^2 x + 2\log_3 x = 3 \Leftrightarrow y^2 + 2y - 3 = 0 \Leftrightarrow (y+3)(y-1) = 0 \Leftrightarrow y_1 = -3 \text{ and } y_2 = 1.$$

We then solve $\log_3 x = y$ for x with these values:

$$y = \log_3 x = -3 \Leftrightarrow x = 3^{-3} \Leftrightarrow \boxed{x = \frac{1}{27}} \text{ and } y = \log_3 x = 1 \Leftrightarrow \boxed{x = 3}.$$

Since both solutions satisfy the existence conditions for the given logarithms, the solution set is $S = \{\frac{1}{27}, 3\}$.

$\log_a x^{f(x)} = f(x) \cdot \log_a x$

b. By the one-to-one property of logarithms, we can take logarithms of both sides to base 3 (notice that both sides are positive):

$$x^{\log_3 x} = 3 \Leftrightarrow \log_3(x^{\log_3 x}) = \log_3 3 \Leftrightarrow (\log_3 x)(\log_3 x) = 1 \Leftrightarrow$$

$$\log_3^2 x = 1 \Leftrightarrow \log_3 x = 1 \text{ or } \log_3 x = -1 \Leftrightarrow \boxed{x = 3} \text{ or } \boxed{x = 3^{-1} = \frac{1}{3}}.$$

When an exponential equation has logarithmic expressions in an exponent, we can take logarithms of both sides to a suitable base in order to get the exponents as factors.

c. Converting the equation into exponential form gives us

$$\log_2(12 - 2^x) = 5 - x \Leftrightarrow 2^{5-x} = 12 - 2^x \Leftrightarrow 2^x + \frac{2^5}{2^x} - 12 = 0.$$

Substituting y for 2^x, we get

$$y + \frac{32}{y} - 12 = 0 \Leftrightarrow y^2 - 12y + 32 = 0 \Leftrightarrow (y-8)(y-4) = 0 \Leftrightarrow y_1 = 8 \text{ and } y_2 = 4.$$

Since both values are positive, we solve

$$y = 2^x = 8 \Leftrightarrow 2^x = 2^3 \Leftrightarrow \boxed{x = 3} \text{ and } y = 2^x = 4 \Leftrightarrow 2^x = 2^2 \Leftrightarrow \boxed{x = 2}.$$

Both values of x satisfy the condition $12 - 2^x > 0$, so the solution set is $S = \{2, 3\}$.

Check Yourself 17

Solve the equations.

a. $\log_2(9-2^x) = 3-x$ b. $x^{\log_3 3x} = 9$ c. $x^{2+\log_2 x} = 256$

d. $\log_2(9^{x-1}+7) = 2+\log_2(3^{x-1}+1)$

Answers

a. $\{0, 3\}$ b. $\{\frac{1}{9}, 3\}$ c. $\{\frac{1}{16}, 4\}$ d. $\{1, 2\}$

4. Equations with Logarithms to Different Bases

In order to work easily with logarithms, we need to have them to the same base. We can use the following properties of logarithms to help us:

For $a, b, c > 0$ and $a, c \neq 1$,

1. $\log_a c = \dfrac{1}{\log_c a}$
2. $\log_{a^n} x = \dfrac{1}{n} \cdot \log_a x$
3. $\log_a b = \dfrac{\log_c b}{\log_c a}$.

EXAMPLE 21 Solve the equations.

a. $\log_2 x + \log_8 x = 8$ b. $(3 \cdot \log_x 4) + (2 \cdot \log_{4x} 4) + (3 \cdot \log_{16x} 4) = 0$

$\log_{a^n} x^m = \dfrac{m}{n}\log_a x$

Solution a. Let us write the second logarithm in base 2:

$$\log_2 x + \log_8 x = 8 \Leftrightarrow \log_2 x + \log_{2^3} x = 8 \Leftrightarrow \log_2 x + \frac{1}{3}\log_2 x = 8 \Leftrightarrow$$

$$(1+\frac{1}{3}) \cdot \log_2 x = 8 \Leftrightarrow \frac{4}{3} \cdot \log_2 x = 8 \Leftrightarrow \log_2 x = 6 \Leftrightarrow x = 2^6 \Leftrightarrow x = 64.$$

Since $x = 64$ satisfies the condition $x > 0$, it is the solution of the given equation.

$\log_a x = \dfrac{1}{\log_x a}$

b. We need to equalize the bases using the properties given above. Since all the logarithms have 4 as their arguments, we can write

$$3 \cdot \log_x 4 + 2 \cdot \log_{4x} 4 + 3 \cdot \log_{16x} 4 = 0 \Leftrightarrow (3 \cdot \frac{1}{\log_4 x}) + (2 \cdot \frac{1}{\log_4 4x}) + (3 \cdot \frac{1}{\log_4 16x}) = 0 \Leftrightarrow$$

$$\frac{3}{\log_4 x} + \frac{2}{\log_4 4 + \log_4 x} + \frac{3}{\log_4 16 + \log_4 x} = 0 \Leftrightarrow \frac{3}{\log_4 x} + \frac{2}{1+\log_4 x} + \frac{3}{2+\log_4 x} = 0.$$

Substituting $y = \log_4 x$, we get

$$\frac{3}{y}+\frac{2}{y+1}+\frac{3}{y+2}=0 \Leftrightarrow 3(y+1)(y+2)+2y(y+2)+3y(y+1)=0 \Leftrightarrow$$
$$8y^2+16y+6=0 \Leftrightarrow 2(2y+3)(2y+1)=0 \Leftrightarrow y_1=-\frac{3}{2} \text{ and } y_2=-\frac{1}{2}.$$

The corresponding values of x are

$$\log_4 x = y = -\frac{3}{2} \Leftrightarrow x = 4^{-\frac{3}{2}} = \frac{1}{\sqrt{4^3}} = \frac{1}{8} \text{ and } \log_4 x = -\frac{1}{2} \Leftrightarrow x = 4^{-\frac{1}{2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

Since both $\boxed{x=\frac{1}{8}}$ and $\boxed{x=\frac{1}{2}}$ satisfy the conditions $x > 0$, $x \neq 1$ $4x \neq 1$, and $16x \neq 1$, the solution set contains both solutions: $S = \{\frac{1}{8}, \frac{1}{2}\}$.

Check Yourself 18

Solve the equations.

a. $2\log_2 x + \log_{\sqrt{2}} x + \log_{1/2} x = 9$ b. $\log_5^2 x + \log_{5x} \frac{5}{x} = 1$

Answers

a. $\{8\}$ b. $\{\frac{1}{25}, 1, 5\}$

5. Logarithmic Equations with a Unique Solution

If we can identify that a given equation has a unique solution (for example: by using the monotony of functions, or by using an arithmetic or geometric argument), and if we observe that a particular value of the variable satisfies the equation, then we do not need to look for another solution.

EXAMPLE 22 Solve the equations.

a. $\log_3 x = 4 - x$ b. $3^{\log_2(x-2)} - 2^{\log_3(x+2)} = 5$

Solution a. We can observe that $x = 3$ satisfies the equation, since $\log_3 3 = 4 - 3 = 1$.

In addition, since the left-hand side of the given equation is strictly increasing and the right-hand side is strictly decreasing, we can conclude that this solution is unique: $\boxed{x = 3}$.

b. Let $\log_2(x-3) = a$ and $\log_3(x+2) = b$. So $3^a - 2^b = 5$.

In addition, we can write

$$\log_2(x-3) = a \Leftrightarrow 2^a = x-3 \Leftrightarrow 2^a + 3 = x \text{ and}$$
$$\log_3(x+2) = b \Leftrightarrow 3^b = x+2 \Leftrightarrow 3^b - 2 = x,$$

which gives us $2^a + 3 = 3^b - 2$, i.e. $3^b - 2^a = 5$.

Since $3^a - 2^b = 5$ and $3^b - 2^a = 5$, we can write

$$3^a - 2^b = 3^b - 2^a \Leftrightarrow 3^a - 3^b + 2^a - 2^b = 0.$$

The only possible solution to this equation is $a = b$, since both $a > b$ and $a < b$ will give a result different from zero. Therefore, we have

$$3^a - 2^a = 5.$$

Obviously, $a = 2$ is a solution. We can also write this equation as

$$1 = 5 \cdot (\frac{1}{3})^a + (\frac{2}{3})^a \qquad \text{(divide all terms by } 3^a\text{)},$$

which leads us to conclude that $3^a - 2^a = 5$ has a unique solution (can you see why?).

Thus $a = b = 2$ and

$$\log_2(x - 3) = a = 2 \Leftrightarrow x - 3 = 2^2 = 4 \Leftrightarrow \boxed{x = 7}.$$

Check Yourself 19

Solve the equations.

a. $x + 2^x + \log_2 x = 7$

b. $9^{\log_3(x-2)} - 5^{\log_5(x+2)} = 4$

Answers

a. $\{2\}$ b. $\{7\}$

6. Logarithmic Equations with Parameters

Sometimes we may be asked to solve an equation for the values of a parameter which satisfy a specific condition, or to investigate the solution(s) of an equation for different values of a parameter.

EXAMPLE 23 Solve $2\log_x a + \log_{ax} a + 3\log_{a^2x} a = 0$ for x in terms of a.

Solution By the existence conditions, $a > 0$, $x > 0$, $x \neq 1$, $ax \neq 1$ and $a^2x \neq 1$.

So we just need to consider positive values of a. Since $a = 1$ is a special case which gives us $(2 \cdot \log_x 1) + (\log_x 1) + (3 \cdot \log_x 1) = 0$, we conclude that any positive value of x different from 1 will satisfy the equation in this case.

For $a > 0$ and $a \neq 1$, if we change the bases to a we get

$$\frac{2}{\log_a x} + \frac{1}{\log_a a + \log_a x} + \frac{3}{\log_a a^2 + \log_a x} = 0 \Leftrightarrow \frac{2}{\log_a x} + \frac{1}{1 + \log_a x} + \frac{3}{2 + \log_a x} = 0.$$

Substituting $\log_a x = y$ gives us

$$\frac{2}{y}+\frac{1}{1+y}+\frac{3}{2+y}=0 \Leftrightarrow \frac{2(y+1)(y+2)+y(y+2)+3y(y+1)}{y(y+1)(y+2)}=0 \Leftrightarrow$$

$$6y^2+11y+4=0 \Leftrightarrow (3y+4)(2y+1)=0 \Leftrightarrow y_1=-\frac{4}{3} \text{ and } y_2=-\frac{1}{2}.$$

The corresponding values of x are

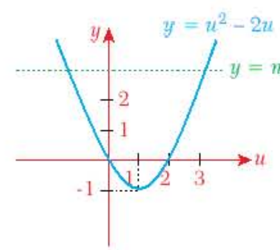
$$\log_a x = y = -\frac{4}{3} \Leftrightarrow x = a^{-\frac{4}{3}} \text{ and } \log_a x = -\frac{1}{2} \Leftrightarrow x = a^{-\frac{1}{2}}.$$

Hence the solution set is $\begin{cases} \varnothing & \text{for } a \le 0 \\ \{a^{-1/2}, a^{-4/3}\} & \text{for } a > 0 \text{ and } a \ne 1 \\ x \in (0, 1) \cup (1, \infty) & \text{for } a = 1. \end{cases}$

EXAMPLE 24 Determine the number of solutions to $(\log_3^2 x) - (2 \cdot \log_3 x) = m$ for different values of m.

Solution Substituting $u = \log_3 x$ gives us $u^2 - 2u = m$.

Let us consider each side of this equation as a function and sketch the graph of each function, as shown opposite.



As we can see, one graph is a parabola and the other is a horizontal line. In addition, the graphs intersect at two points when $m > -1$ and at one point when $m = -1$, and they do not intersect when $m < -1$. In other words, the equation $u^2 - 2u = m$ has two solutions for $m > -1$, one solution for $m = -1$ and no solutions for $m < -1$. Since $u = \log_3 x$ is a one-to-one function, it has the same number of solutions for the different values of m. Hence we conclude that the given equation has

$$\begin{cases} \text{two solutions for } m > -1 \\ \text{one solution for } m = -1 \\ \text{no solution for } m < -1. \end{cases}$$

Check Yourself 20

1. Solve $\log_a x + \log_{\sqrt{a}} x + \log_{\sqrt[3]{a^2}} x = 27$ for x in terms of a.

2. Determine the number of solutions to $6\log_2 x - \log_2^2 x - 8 = a$ for different values of a.

Answers

1. $S = \begin{cases} \varnothing & \text{if } \ a \le 0 \text{ or } a = 1 \\ \{a^6\} & \text{if } \ a > 0 \text{ and } a \ne 1 \end{cases}$

2. $\begin{cases} \text{no solution} & \text{if } \ a > 1 \\ \text{one solution} & \text{if } \ a = 1 \\ \text{two solutions} & \text{if } \ a < 1 \end{cases}$

B. LOGARITHMIC INEQUALITIES

Sometimes we are asked to solve inequalities between two functions of the form $\log_{f(x)} g(x)$ and $\log_{f(x)} h(x)$, for example: $\log_{f(x)} g(x) > \log_{f(x)} h(x)$, $\log_{f(x)} g(x) \le \log_{f(x)} h(x)$, etc. In this case we can establish the systems

$\begin{cases} f(x) > 1 \\ g(x) < h(x) \end{cases}$ or $\begin{cases} 0 < f(x) < 1 \\ g(x) > h(x) \end{cases}$ to respect the monotone property of logarithmic functions and combine them with the existence conditions

$$\begin{cases} f(x) > 0, \ f(x) \neq 1 \\ g(x) > 0 \\ h(x) > 0. \end{cases}$$

As a result, we get

$\begin{cases} f(x) > 1 \\ g(x) < h(x) \\ g(x) > 0 \end{cases}$ and $\begin{cases} f(x) > 0 \\ f(x) < 1 \\ g(x) > h(x) \\ h(x) > 0 \end{cases}$ as possible systems for the inequality.

We can solve the resulting systems using a sign table. The values satisfying any of these systems will be included in the solution set of the original inequality.

For logarithmic inequalities which do not have functions of the form $\log_{f(x)} g(x)$ and $\log_{f(x)} h(x)$, we first apply the properties of logarithms in order to write them in this form.

Let us look at some examples.

EXAMPLE 25 Solve the inequalities.

a. $\log_3(3 - x) < \log_3(x + 5)$

b. $\log_{\frac{1}{5}}(x-2) \ge \log_{\frac{1}{5}}(1-2x)$

c. $\log_2(x^2 + 4x + 3) \le 3$

d. $\log_{\frac{1}{4}} \dfrac{x-3}{x+3} \ge -\dfrac{1}{2}$

e. $\log(x^2 - 5x + 7) < 0$

f. $\log_{2x}(5x - 3) \le 1$

Solution a. Using the monotone property of logarithmic functions and the existence conditions for logarithms, we obtain the system

$$\begin{cases} 3 - x < x + 5 \\ 3 - x > 0 \end{cases} \Leftrightarrow \begin{cases} -1 < x \\ 3 > x \end{cases} \Leftrightarrow \boxed{x \in (-1, 3)}.$$

b. Since the base is between zero and 1, we establish the system

$$\begin{cases} x - 2 \le 1 - 2x \\ x - 2 > 0 \\ 1 - 2x > 0 \end{cases} \Leftrightarrow \begin{cases} x \le 1 \\ x > 2 \\ \dfrac{1}{2} > x \end{cases}$$, which has no solution.

c. The right-hand side of the inequality is not in logarithmic form. However, we can use the identity $\log_2 2 = 1$ and write

$$\log_2(x^2 + 4x + 3) \le 3 \cdot \log_2 2 \Leftrightarrow \log_2(x^2 + 4x + 3) \le \log_2 2^3.$$

Since the base is greater than 1, this inequality is equivalent to

$$\begin{cases} x^2 + 4x + 3 \le 2^3 \\ x^2 + 4x + 3 > 0 \end{cases} \Leftrightarrow \begin{cases} (x + 5) \cdot (x - 1) \le 0 \\ (x + 3) \cdot (x + 1) > 0 \end{cases} \Leftrightarrow \begin{cases} x \in [-5, 1] \\ x \in (-\infty, -3) \cup (-1, \infty). \end{cases}$$

So the common solution is $x \in [-5, -3) \cup (-1, 1]$.

d. Using a similar approach, we can write

$$\log_{\frac{1}{4}} \frac{x-3}{x+3} \ge -\frac{1}{2} \Leftrightarrow \log_{\frac{1}{4}} \frac{x-3}{x+3} \ge \log_{\frac{1}{4}} \left(\frac{1}{4}\right)^{-\frac{1}{2}} \Leftrightarrow \log_{\frac{1}{4}} \frac{x-3}{x+3} \ge \log_{\frac{1}{4}} 2.$$

Therefore we solve the system

$$\begin{cases} \dfrac{x-3}{x+3} \le 2 \\ \dfrac{x-3}{x+3} > 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{-x-9}{x+3} \le 0 \\ \dfrac{x-3}{x+3} > 0. \end{cases}$$

By considering the critical values, we establish the sign table:

x	$-\infty$		-9		-3		3		∞
$\dfrac{x-3}{x+3}$		$+$		$+$	$\circ$	$-$	$\circ$	$+$	
$\dfrac{-x-9}{x+3}$		$-$	$\bullet$	$+$	$\circ$	$-$		$-$	
Common solution			$\bullet$				$\circ$		

Therefore the solution is $x \in (-\infty, -9] \cup (3, \infty)$.

Note

For inequalities of the form $\log_a g(x) > b$ or $\log_a g(x) < b$, we can use the property $b = \log_a a^b$ and write

$$\text{for } a > 1: \begin{cases} \log_a g(x) < b \Leftrightarrow 0 < g(x) < a^b \\ \log_a g(x) > b \Leftrightarrow g(x) > a^b, \end{cases}$$

$$\text{for } a \in (0, 1): \begin{cases} \log_a g(x) < b \Leftrightarrow g(x) > a^b \\ \log_a g(x) > b \Leftrightarrow 0 < g(x) < a^b. \end{cases}$$

e. We have $\log(x^2 - 5x + 7) < 0 \Leftrightarrow \log(x^2 - 5x + 7) < \log 1$.

Establishing the appropriate system, we get

$$\begin{cases} x^2 - 5x + 7 < 1 \\ x^2 - 5x + 7 > 0 \end{cases} \Leftrightarrow \begin{cases} x^2 - 5x + 6 < 0 \\ x \in \mathbb{R} \end{cases} \Leftrightarrow \begin{cases} (x - 2)(x - 3) < 0 \\ x \in \mathbb{R} \end{cases} \Leftrightarrow x \in (2, 3).$$

$\log_a a = 1$

f. By the properties of logarithms, we can write the inequality as

$$\log_{2x}(5x-3) \le 1 \Leftrightarrow \log_{2x}(5x-3) \le \log_{2x}(2x).$$

There are two cases: the base may be between zero and 1, or it may be greater than 1. Accordingly we have the two systems

$$\begin{cases} 0 < 2x < 1 \\ 5x - 3 \ge 2x \end{cases} \text{ and } \begin{cases} 2x > 1 \\ 5x - 3 \le 2x \\ 5x - 3 > 0. \end{cases}$$

The first system has no solution, because we obtain $\begin{cases} 0 < x < \dfrac{1}{2} \\ x \ge 1 \end{cases}$, which is impossible.

For the second system, we have

$$\begin{cases} x > \dfrac{1}{2} \\ x \le 1 \\ x > \dfrac{3}{5} \end{cases}$$, which has the solution $x \in (\frac{3}{5}, 1]$. So this is the solution to the inequality.

EXAMPLE 26 Solve the inequalities.

a. $\log_{\frac{1}{2}}\left(\log_6 \dfrac{x^2+x}{x+4}\right) < 0$ b. $(\log_x 2 \cdot \log_{2x} 2) > \log_{4x} 2$

Solution a. We establish the system

$$\begin{cases} \log_6 \dfrac{x^2+x}{x+4} > 1 \\ \dfrac{x^2+x}{x+4} > 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{x^2+x}{x+4} > 6 \\ \dfrac{x^2+x}{x+4} > 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{x^2+x}{x+4} - 6 > 0 \\ \dfrac{x^2+x}{x+4} > 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{x^2-5x-24}{x+4} > 0 \\ \dfrac{x^2+x}{x+4} > 0. \end{cases}$$

Considering the critical values –4, –3, 8 for $\dfrac{x^2-5x-24}{x+4}$, and –4, –1, 0 for $\dfrac{x^2+x}{x+4}$, we establish the following sign table:

x	$-\infty$ … -4	-4 … -3	-3 … -1	-1 … 0	0 … 8	8 … ∞
$\dfrac{x^2-5x-24}{x+4}$	–	+	–	–	–	+
$\dfrac{x^2+x}{x+4}$	–	+	+	–	+	+
Common solution						

Therefore the solution is $x \in (-4, -3) \cup (8, \infty)$.

b. Using the rule $\log_a b = \dfrac{1}{\log_b a}$, we get

$$\left(\frac{1}{\log_2 x} \cdot \frac{1}{\log_2 2x}\right) > \frac{1}{\log_2 4x} \Leftrightarrow \left(\frac{1}{\log_2 x} \cdot \frac{1}{1+\log_2 x}\right) > \frac{1}{2+\log_2 x}.$$

Substituting y for $\log_2 x$, we get

$$\frac{1}{y\cdot(y+1)} > \frac{1}{y+2} \Leftrightarrow \frac{1}{y\cdot(y+1)} - \frac{1}{y+2} > 0 \Leftrightarrow \frac{-y^2+2}{y\cdot(y+1)\cdot(y+2)} > 0.$$

We can establish the corresponding sign table as

y	$-\infty$ … -2	-2 … $-\sqrt{2}$	$-\sqrt{2}$ … -1	-1 … 0	0 … $\sqrt{2}$	$\sqrt{2}$ … ∞
$\dfrac{-y^2+2}{y\cdot(y+1)\cdot(y+2)}$	+	–	+	–	+	–

which leads us to the solution $y \in (-\infty, -2) \cup (-\sqrt{2}, -1) \cup (0, \sqrt{2})$.

We can find the corresponding values of x by considering the three cases:

For $y \in (-\infty, -2)$, $\quad y = \log_2 x < -2 \Leftrightarrow 0 < x < 2^{-2} \Leftrightarrow 0 < x < \dfrac{1}{4}$.

For $y \in (-\sqrt{2}, -1)$, $\quad -\sqrt{2} < \log_2 x < -1 \Leftrightarrow 2^{-\sqrt{2}} < x < 2^{-1} \Leftrightarrow \dfrac{1}{2^{\sqrt{2}}} < x < \dfrac{1}{2}$.

For $y \in (0, \sqrt{2})$, $\quad 0 < \log_2 x < \sqrt{2} \Leftrightarrow 1 < x < 2^{\sqrt{2}}$.

Therefore the solution is $x \in (0, \dfrac{1}{4}) \cup (\dfrac{1}{2^{\sqrt{2}}}, \dfrac{1}{2}) \cup (1, 2^{\sqrt{2}})$.

Check Yourself 21

Solve the inequalities.

a. $\log_3(1-2x) \geq \log_3(5x-2)$

b. $\log_{\frac{1}{2}}(3^{x+1}-3) > \log_{\frac{1}{2}}(3^x+1)$

c. $\log_2(2x-1) > 0$

d. $\log_{\frac{1}{3}}(5x-1) \geq 0$

e. $\dfrac{\log_2(x+1)}{x+1} < 0$

f. $2 \cdot \log_8(x-2) - \log_8(x+4) > 1$

g. $1 - 2\cdot\log_{\frac{1}{9}}(x+1) > \log_3(x-3)$

h. $\log_{x+1}(3x-1) > 1$

Answers

a. $x \in (\dfrac{2}{5}, \dfrac{3}{7})$ b. $x \in (0, \log_3 2)$ c. $x \in (1, \infty)$ d. $x \in (\dfrac{1}{5}, \dfrac{2}{5}]$

e. $x \in (-1, 0)$ f. $x \in (14, \infty)$ g. $x \in (3, \infty)$ h. $x \in (-1, 0) \cup (1, \infty)$

EXERCISES 3.2

A. Logarithmic Equations

1. Solve the equations.

a. $\log_2(3x - 4) = 3$

b. $\log_3 x + \log_3(x + 6) = 3$

c. $\log(x + 2) - \log(x - 2) = 0$

d. $\log_9 x^2 - \log_9(2x + 5) - \log_9(2x - 5) = -\frac{1}{2}$

e. $\log_7(2x^2 - 5x) = 1$

f. $\log_2(\log_4 x) = 2$

g. $\log_4(\log_2(\log_3(2x-1))) = \frac{1}{2}$

2. Solve the equations.

a. $\log_{\frac{1}{5}}(x^2 + 3x - 4) = \log_{\frac{1}{5}}(2x+2)$

b. $\log_3 \frac{2x^2 - 54}{x+3} = \log_3(x-4)$

c. $\log_4(3^x + 1) = \log_4(-2^x + 14)$

d. $\log_{5x-2}(2x^2) = \log_{5x-2}(x + 1)$

3. Solve the equations.

a. $e^{\ln(2x-1)} - 10^{\log x} = 1$

b. $3\log_3^2 x + \log_3 x^2 = 5$

c. $\log_2 (2^x - 3) + x = 2$

d. $x + \log_3 (3^x - 8) = 2$

e. $x^{\ln x} = x$

f. $3^{\ln x} + x^{\ln 3} = 54$

g. $\log_x|2(1 - x)| - \log_{1-x}(2x) = 0$

h. $\log_3(2^x + 1) = \log_2(3^x - 1)$

4. Solve the equations.

a. $\log_2 x + \log_4 x + \log_{16} x = 7$

b. $\log_x 4 + \log_{x^2} 64 = 5$

c. $(3 \cdot \log_x 16) - (4 \cdot \log_{16} x) = 2 \cdot \log_2 x$

d. $x^{\log_3 x} + 3^{\log_3^2 x} = 162$

B. Logarithmic Inequalities

5. Find the solution set for each inequality, as an interval.

a. $\log_4(2x - 5) < 1$

b. $\ln(x + 2) \geq 0$

c. $\log_{3/4}(2x - 4) > 2$

d. $\log_4(x^2 - 9) \leq 2$

e. $\log_{\frac{3}{5}}(x^2 - 3) > 0$

f. $\ln x \leq 0$

g. $\log_{0.1}(x^2 + x + 1) - \log_{0.1} 3 > 0$

h. $\log_3(x - 1) - \log_3 5 < 1$

i. $\log_2(\log (x - 3)) \leq 0$

j. $\log_{\frac{1}{2}}(\log_5(x-6)) \geq 0$

k. $\log_2(x - 1) + \log_2(x - 3) < 3$

6. Solve $\frac{1+\log_a^2 x}{1+\log_a x} > 1$ for $0 < a < 1$.

3 SYSTEMS OF EQUATIONS AND INEQUALITIES

A. SYSTEMS OF EQUATIONS

We can solve systems of exponential or logarithmic equations by using the properties we have studied so far combined with algebraic operations, elimination and substitution. We write the solution of a system as a set of ordered pairs, triples or quadruples, etc. depending on the number of unknowns in the system.

EXAMPLE 27 Solve the systems of equations.

a. $\begin{cases} 3^x + 3^y = 28 \\ 3^{x+y} = 27 \end{cases}$ b. $\begin{cases} 2^x \cdot 3^y = 12 \\ 2^y \cdot 3^x = 18 \end{cases}$ c. $\begin{cases} x^{y+1} = 27 \\ x^{2y-5} = \dfrac{1}{3} \end{cases}$

Solution a. Substituting $3^x = a > 0$ and $3^y = b > 0$, we have

$$\begin{cases} a + b = 28 \\ a \cdot b = 27 \quad (3^{x+y} = 3^x \cdot 3^y) \end{cases} \Leftrightarrow \begin{cases} a = 28 - b \\ a \cdot b = 27 \end{cases} \Leftrightarrow \begin{cases} a = 28 - b \\ (28 - b) \cdot b = 27 \end{cases} \Leftrightarrow$$

$$\begin{cases} a = 28 - b \\ b^2 - 28b + 27 = 0 \end{cases} \Leftrightarrow \begin{cases} a = 28 - b \\ (b - 27)(b - 1) = 0 \end{cases} \Leftrightarrow \begin{cases} a = 28 - b \\ b = 1 \text{ or } b = 27. \end{cases}$$

So the solutions for a and b are $\begin{cases} b = 1 \\ a = 27 \end{cases}$ or $\begin{cases} b = 27 \\ a = 1 \end{cases}$, which give us $\begin{cases} 3^x = 1 \\ 3^y = 27 \end{cases}$ and $\begin{cases} 3^x = 27 \\ 3^y = 1 \end{cases}$.

Hence the possible cases are $\begin{cases} x = 0 \\ y = 3 \end{cases}$ or $\begin{cases} x = 3 \\ y = 0 \end{cases}$.

So the solution is $\{(0, 3), (3, 0)\}$.

b. We begin by multiplying the second equation by the first:

$$\begin{cases} 2^x \cdot 3^y = 12 \\ 2^x \cdot 3^y \cdot 2^y \cdot 3^x = 12 \cdot 18 \end{cases} \Leftrightarrow \begin{cases} 2^x \cdot 3^y = 12 \\ 2^{x+y} \cdot 3^{x+y} = 216 \end{cases} \Leftrightarrow \begin{cases} 2^x \cdot 3^y = 12 \\ 6^{x+y} = 6^3 \end{cases} \Leftrightarrow \begin{cases} 2^x \cdot 3^y = 12 \\ x + y = 3 \end{cases} \Leftrightarrow$$

$$\begin{cases} 2^x \cdot 3^{3-x} = 12 \\ y = 3 - x \end{cases} \Leftrightarrow \begin{cases} 2^x \cdot \dfrac{27}{3^x} = 12 \\ y = 3 - x \end{cases} \Leftrightarrow \begin{cases} (\dfrac{2}{3})^x = (\dfrac{2}{3})^2 \\ y = 3 - x \end{cases} \Leftrightarrow \begin{cases} x = 2 \\ y = 1. \end{cases}$$

So $\{(2, 1)\}$ is the solution.

c. Taking the square of the first equation, we have

$$\begin{cases} (x^{y+1})^2 = (27)^2 \\ x^{2y-5} = \dfrac{1}{3} \end{cases} \Leftrightarrow \begin{cases} x^{2y} \cdot x^2 = 3^6 \\ x^{2y} \cdot x^{-5} = 3^{-1} \end{cases} \Leftrightarrow \begin{cases} 3^{-1} \cdot x^5 \cdot x^2 = 3^6 \\ x^{2y} = 3^{-1} \cdot x^5 \end{cases} \Leftrightarrow$$

$$\begin{cases} x^7 = 3^7 \\ x^{2y} = 3^{-1} \cdot x^5 \end{cases} \Leftrightarrow \begin{cases} x = 3 \\ 3^{2y} = 3^{-1} \cdot 3^5 \end{cases} \Leftrightarrow \begin{cases} x = 3 \\ 3^{2y} = 3^4 \end{cases} \Leftrightarrow \begin{cases} x = 3 \\ y = 2. \end{cases}$$

Hence $\{(3, 2)\}$ is the solution.

EXAMPLE 28 Solve the systems of equations.

a. $\begin{cases} x + y = 7 \\ \ln x + \ln y = 2\ln 2 + \ln 3 \end{cases}$

b. $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} = \dfrac{2}{15} \\ \log_3 x + \log_3 y = 1 + \log_3 5 \end{cases}$

Solution a. We can use the properties of logarithms to write

$$\ln x + \ln y = 2\ln 2 + \ln 3 \Leftrightarrow \ln(x \cdot y) = \ln(2^2 \cdot 3) \Leftrightarrow x \cdot y = 12.$$

So we just need to solve the system

$$\begin{cases} x + y = 7 \\ x \cdot y = 12 \end{cases} \Leftrightarrow \begin{cases} y = 7 - x \\ x \cdot (7 - x) = 12 \end{cases} \Leftrightarrow \begin{cases} y = 7 - x \\ x^2 - 7x + 12 = 0 \end{cases} \Leftrightarrow \begin{cases} y = 7 - x \\ x = 3 \text{ or } x = 4. \end{cases}$$

Hence the solution is $\{(4, 3), (3, 4)\}$.

Notice that both pairs satisfy the existence conditions for logarithms.

b. Using a similar approach, we can work on the second equation in the system to get

$$\log_3 x + \log_3 y = 1 + \log_3 5 \Leftrightarrow \log_3(x \cdot y) = \log_3(3 \cdot 5) \Leftrightarrow x \cdot y = 15.$$

So we have the system

$$\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} = \dfrac{2}{15} \\ x \cdot y = 15 \end{cases} \Leftrightarrow \begin{cases} \dfrac{1}{x} - \dfrac{x}{15} = \dfrac{2}{15} \\ y = \dfrac{15}{x} \end{cases} \Leftrightarrow \begin{cases} \dfrac{15 - x^2}{15x} = \dfrac{2}{15} \\ y = \dfrac{15}{x}. \end{cases} \Leftrightarrow$$

$$\begin{cases} x^2 + 2x - 15 = 0 \\ y = \dfrac{15}{x} \end{cases} \Leftrightarrow \begin{cases} x = -5 \text{ or } x = 3 \\ y = \dfrac{15}{x} \end{cases} \Leftrightarrow \begin{cases} x = -5 \\ y = -3 \end{cases} \Leftrightarrow \text{ or } \begin{cases} x = 3 \\ y = 5 \end{cases}$$

To satisfy the existence conditions, x and y must be positive. Therefore we eliminate the case $x = -5$ and write the solution as $\{(3, 5)\}$.

Check Yourself 22

Solve the systems of equations.

1. a. $\begin{cases} 3^x + 3^y = 4 \\ x + y = 1 \end{cases}$ b. $\begin{cases} 2^x \cdot 3^y = 36 \\ 2^x + 3^y = 13 \end{cases}$ c. $\begin{cases} 2^x \cdot 3^y = 24 \\ 2^y \cdot 3^x = 54 \end{cases}$

2. a. $\begin{cases} 5x + 4y = 7 \\ \ln x + \ln y = \ln(\dfrac{3}{5}) \end{cases}$ b. $\begin{cases} 4x^2 - y^2 = 2 \\ \log_2(2x + y) - \log_2(2x - y) = 1 \end{cases}$

c. $\begin{cases} \log_y x - \log_x y = \dfrac{8}{3} \\ x = 16y \end{cases}$ d. $\begin{cases} y \cdot x^{\log_y x} = x^2\sqrt{x} \\ \log_3 y \cdot \log_y(y - 2x) = 1 \end{cases}$

Answers

1. a. $\{(0, 1), (1, 0)\}$ b. $\{(\log_2 9, \log_3 4), (2, 2)\}$ c. $\{(3, 1)\}$

2. a. $\{(\frac{4}{5}, \frac{3}{4}), (\frac{3}{5}, 1)\}$ b. $\{(\frac{3}{4}, \frac{1}{2})\}$ c. $\{(2, \frac{1}{8}), (64, 4)\}$ d. $\{(3, 9)\}$

B. SYSTEMS OF INEQUALITIES

To solve a system of exponential or logarithmic inequalities, we consider each inequality separately and find its solution. Then we add the existence conditions of logarithms to the system and find a common solution set. We can also use a sign table to look at the different cases.

The following examples illustrate this approach.

EXAMPLE 29 Solve the systems of inequalities.

a. $\begin{cases} (\frac{1}{8})^{-x} < 128 \\ 2^{4x} \geq 16 \end{cases}$ b. $\begin{cases} 3^{2x+1} - 3^{x+2} + 6 > 0 \\ 3^{2x+2} - (2 \cdot 3^{x+2}) - 27 < 0 \end{cases}$

Solution a. $\begin{cases} 2^{3x} < 2^7 \\ 2^{4x} \geq 2^4 \end{cases} \Leftrightarrow \begin{cases} 3x < 7 \\ 4x \geq 4 \end{cases} \Leftrightarrow \begin{cases} x < \frac{7}{3} \\ x \geq 1 \end{cases}$

Hence the solution set is $S = (-\infty, \frac{7}{3}) \cap [1, \infty) = [1, \frac{7}{3})$.

b. Using the rules of exponents, we get

$$\begin{cases} 3^{2x+1} - 3^{x+2} + 6 > 0 \\ 3^{2x+2} - (2 \cdot 3^{x+2}) - 27 < 0 \end{cases} \Leftrightarrow \begin{cases} (3 \cdot (3^x)^2) - 9 \cdot 3^x + 6 > 0 \\ (9 \cdot (3^x)^2) - (18 \cdot 3^x) - 27 < 0 \end{cases} \Leftrightarrow \begin{cases} 3 \cdot (3^x - 2) \cdot (3^x - 1) > 0 \\ 9 \cdot (3^x - 3) \cdot (3^x + 1) < 0. \end{cases}$$

Let us find the critical values and establish a sign table:

$3^x - 2 = 0 \Leftrightarrow 3^x = 2 \Leftrightarrow x = \log_3 2$

$3^x - 1 = 0 \Leftrightarrow 3^x = 1 \Leftrightarrow x = 0$

$3^x - 3 = 0 \Leftrightarrow 3^x = 3 \Leftrightarrow x = 1$

$3^x + 1 > 0$, so there is no critical value here.

x	$-\infty$		0		$\log_3 2$		1		∞
$3^{2x+1} - 3^{x+2} + 6$		$+$	0	$-$	0	$+$		$+$	
$3^{2x+2} - (2 \cdot 3^{x+2}) - 27$		$-$		$-$		$-$	0	$+$	
Common solution									

In conclusion, the solution set is $S = (-\infty, 0) \cup (\log_3 2, 1)$.

EXAMPLE 30 Solve the systems of inequalities.

a. $\begin{cases} \log_2(3 - x) < 2 \\ \log_{\frac{1}{3}}(x + 2) < 0 \end{cases}$ b. $\begin{cases} \log_{\frac{1}{3}} x + 2 \cdot \log_{\frac{1}{9}}(x - 1) \leq \log_{\frac{1}{3}} 6 \\ \log_2(x + 1) < 1 - 2 \log_4 x \end{cases}$

Solution a. Adding the existence conditions, the system becomes

$$\begin{cases} \log_2(3-x) < 2 \\ \log_{\frac{1}{3}}(x+2) < 0 \\ 3-x > 0 \\ x+2 > 0 \end{cases} \Leftrightarrow \begin{cases} 3-x < 2^2 \\ x+2 > (\frac{1}{3})^0 \\ 3 > x \\ x > -2 \end{cases} \Leftrightarrow \begin{cases} -1 < x \\ x > -1 \\ 3 > x \\ x > -2. \end{cases}$$

Hence the common solution is $\{(-1, 3)\}$.

b. By working on the logarithmic expressions and introducing the existence conditions, we get

$$\begin{cases} \log_{\frac{1}{3}} x + 2 \cdot \log_{(\frac{1}{3})^2}(x-1) \le \log_{\frac{1}{3}} 6 \\ \log_2(x+1) < 1 - 2\log_{2^2} x \\ x > 0 \\ x-1 > 0 \\ x+1 > 0 \end{cases} \Leftrightarrow \begin{cases} \log_{\frac{1}{3}} x + \log_{\frac{1}{3}}(x-1) \le \log_{\frac{1}{3}} 6 \\ \log_2(x+1) + \log_2 x < 1 \\ x > 0 \\ x > 1 \\ x > -1 \end{cases} \Leftrightarrow$$

$$\begin{cases} \log_{\frac{1}{3}}|x \cdot (x-1)| \le \log_{\frac{1}{3}} 6 \\ \log_2|(x+1) \cdot x| < \log_2 2 \\ x > 1 \end{cases} \Leftrightarrow \begin{cases} x^2 - x \ge 6 \\ x^2 + x < 2 \\ x > 1 \end{cases} \Leftrightarrow \begin{cases} x^2 - x - 6 \ge 0 \\ x^2 + x - 2 < 0 \\ x > 1 \end{cases} \Leftrightarrow$$

$$\begin{cases} (x-3)(x+2) \ge 0 \\ (x+2)(x-1) < 0 \\ x > 1 \end{cases} \Leftrightarrow \begin{cases} x \le -2 \text{ or } x \ge 3 \\ -2 < x < 1 \\ x > 1. \end{cases}$$

Since there is no common solution for these intervals, we conclude that the system has no solution.

Check Yourself 23

Solve each system of inequalities.

1. a. $\begin{cases} 3^{2x} \le 243 \\ (\frac{1}{9})^{2x} < 27 \end{cases}$ b. $\begin{cases} 2^x + 2^{x+1} + 2^{x+2} \ge 28 \\ 5^{x+1} + 5^x < 750 \end{cases}$ c. $\begin{cases} 9^x - 3^{x+2} < 3^{x-2} - 1 \\ x^3 + 2^x \ge 3 \end{cases}$

2. a. $\begin{cases} \log_3(2x+1) < 2 \\ \log_{\frac{1}{3}}(3x-1) \ge 1 \end{cases}$ b. $\begin{cases} \log_3 \dfrac{2x+1}{x+1} < 1 \\ \log_{\frac{1}{4}} \dfrac{x-3}{x+3} < -\dfrac{1}{2} \end{cases}$

Answers

1. a. $(-\frac{3}{4}, \frac{5}{2}]$ b. $[2, 3)$ c. $[1, 2)$

2. a. $(\frac{1}{3}, \frac{4}{9}]$ b. $(-9, -3)$

EXERCISES 3.3

A. Systems of Equations

1. Find the ordered pair(s) (x, y) that satisfy each system.

a. $\begin{cases} 5^{3x} = 5^{4y+7} \\ 2^x \cdot 4^y = 16 \end{cases}$ b. $\begin{cases} 4^{(x-y)^2-1} = 1 \\ 5^{x+y} = 125 \end{cases}$

c. $\begin{cases} 4^{x+y} = 128 \\ 5^{3x-2y-3} = 1 \end{cases}$ d. $\begin{cases} 3975^{x^2-y^2-4} = 1 \\ 2^{x+y+1} = \dfrac{1}{8} \end{cases}$

e. $\begin{cases} 2^{2x} - 2^x - 2 = 0 \\ 9^{xy} - 1 = 0 \end{cases}$ f. $\begin{cases} 3^x \cdot 5^y = 45 \\ 3^y \cdot 5^x = 75 \end{cases}$

2. Solve the systems of equations.

a. $\begin{cases} 27^{2x} + 125^{y/3} = 8 \\ 3^{12x} - 5^{2y} = -16 \end{cases}$ b. $\begin{cases} 3^{\frac{x-y}{2}} + (3^x \,.\, 3^{-y}) = 12 \\ 3^x + 3^{-y} = 10 \end{cases}$

c. $\begin{cases} (3 \cdot 2^x) + (2 \cdot 3^y) = 2.75 \\ 2^x - 3^y = -0.75 \end{cases}$

d. $\begin{cases} 3^x + (3 \cdot 5^{x+y}) = 378 \\ (5 \cdot 3^x) + 5^{x+y+1} = 640 \end{cases}$

✪e. $\begin{cases} 5^{x+2y} + 9^{x+2y} = 14 \\ 3^{2x-y} + 5^{2x-y} = 8 \end{cases}$

✪f. $\begin{cases} 5^{x+2y} + 2 = 7^{x+2y} \\ 3^{x-2y} + 98 = 5^{x-2y} \end{cases}$

3. Solve the systems of equations.

a. $\begin{cases} \log_2 x + 2\log_2 y = 3 \\ x^2 + y^4 = 16 \end{cases}$ b. $\begin{cases} 3^x \cdot 2^y = 972 \\ \log_{\sqrt{3}}(x - y) = 2 \end{cases}$

c. $\begin{cases} 3^x \cdot 2^y = 576 \\ \log_{\sqrt{2}}(y - x) = 4 \end{cases}$ d. $\begin{cases} x + y = 5 \\ \log_2 x + \log_2 y = 2 \end{cases}$

e. $\begin{cases} x^2 + y^2 = 90 \\ \log_3 x + \log_3 y = 3 \end{cases}$

4. Solve the systems of equations.

a. $\begin{cases} \log_4 x - \log_x y = \dfrac{7}{6} \\ x \cdot y = 16 \end{cases}$ b. $\begin{cases} y - \log_3 x = 1 \\ x^4 = 3^{12} \end{cases}$

c. $\begin{cases} \log_y x + \log_x y = \dfrac{5}{2} \\ x + y = 20 \end{cases}$ d. $\begin{cases} \log_y x + \log_x y = 2 \\ x^2 + y = 30 \end{cases}$

e. $\begin{cases} 2(\log_y x + \log_x y) = 5 \\ x \cdot y = 8 \end{cases}$

B. Systems of Inequalities

5. Solve each system of inequalities.

a. $\begin{cases} (\dfrac{2}{3})^x \cdot (\dfrac{8}{9})^{-x} > \dfrac{27}{64} \\ 2^{x^2-6x-3} < 16 \end{cases}$

b. $\begin{cases} 25^x - (6 \cdot 5^x) < 5 \\ \log_{2x-3} 9 < 2 \end{cases}$

c. $\begin{cases} 6^{x+1} + 3^{x+1} > 3^{x+2} - 6^x + 3^x \\ 16\sqrt{4^{\frac{x}{4}-5}} > 2^{\sqrt{x}+1} \end{cases}$

d. $\begin{cases} 3^{x+1} + 3^{x+2} + 3^{x+3} > 5^{x+2} + (3 \cdot 5^{x+1}) - 5^x \\ 2^2 + 2x \le 0 \end{cases}$

e. $\begin{cases} 3^{2x-2} + 27 \le 12 \cdot 3^{x-1} \\ (3 + 2\sqrt{2})^x + (3 - 2\sqrt{2})^x > 34 \end{cases}$

THE SLIDE RULE

Within 20 years of the publication of Henry Briggs' tables of logarithms, the use of logarithms had spread all the way around the world. Logarithms were no longer a strange mathematical technique used only by great scientists like Johannes Kepler: they had become a common tool in schoolrooms. Logarithms were also commonly used in all trades and professions that needed to make frequent calculations. It is hard to imagine an invention which has helped the process of computation more dramatically, the only exceptions being the modern digital calculator and the computer.

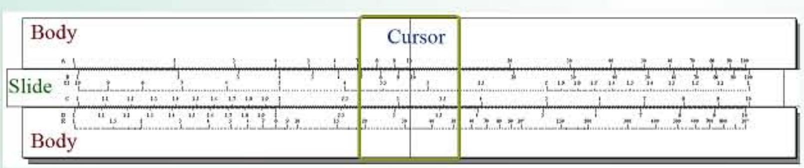
At about this time, Briggs' colleague Edmund Gunter realized that instead of carrying around a set of logarithm tables, he could inscribe distances on a piece of wood to represent the logarithms. A pair of dividers could be used to measure these lengths, and adding the lengths together would be equivalent to adding the logarithms. This device (known as *Gunter's Line of Numbers*) quickly became a popular calculating tool.

Of course, it did not take long for William Oughtred to see that in order to measure these distances with a pair of dividers, it is only necessary to have two scales which can slide past one another. This was the basic principle behind a calculating tool called the slide rule. In 1850, a French army officer called Amedee Mannheim took Oughtred's idea and invented the slide rule that we know today.

An advertisement from the days of the slide rule

A slide rule is a computing tool which uses the properties of logarithms to allow the user to perform complicated calculations easily. Since $\log(ab) = \log a + \log b$, it is quite easy to understand how a slide rule works.

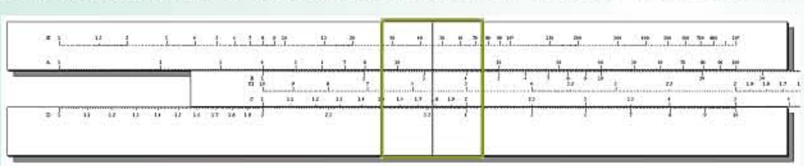
A slide rule consists of three main parts, as shown in the picture:



The body and slide of the rule are printed with different scales, and each scale has a specific function. The letters to the left and right of the individual scales are the names of the scales: A & B, CI & C, D, K, etc. In its most basic form, the slide rule uses two logarithmic scales which allow the user to rapidly multiply and divide large numbers. More complex slide rules allow the user to calculate other things such as square roots, exponential expressions, logarithms and trigonometric functions.

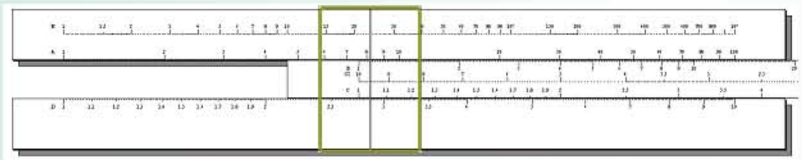
To make a calculation on a slide rule, we use one hand to hold the body of the rule and the other to move the slide and cursor to align the different scales. The numbers which are brought into line show the approximate value of the product, quotient or other desired result.

For example, let us begin with a simple product: multiply 2 by 4. We move the left index of the C scale over 2, then move the cursor to 4 on the C scale and finally read the answer 8 on the D scale. Physically, we have added the logarithm of 2 to the logarithm of 4, but we have in fact multiplied the the two numbers.



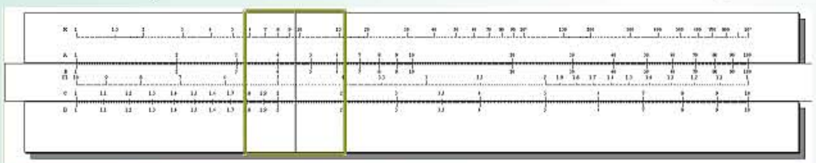
Notice that on a slide rule, multiplying 2 by 4 is functionally equivalent to multiplying 2 by 400, or 200 by 4 000 000. It is up to you to work out the powers of 10 in each case.

For this reason, it is best to express the numbers in scientific notation before using the slide rule. For example, we can write 200 times 4 000 000 as $2.0 \times 10^2 \times 4.0 \times 10^6$, which reduces to $2.0 \times 4.0 \times 10^8$, and finally 8×10^8.

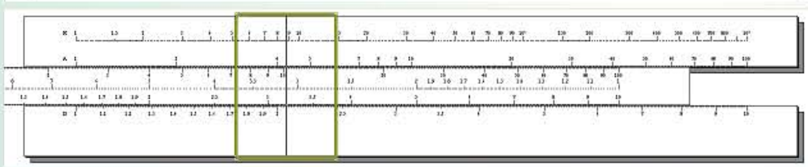


To divide two numbers, we subtract the logarithm of the second number from the first. For example, to divide 5.5 by 2, we line up 2 on the C scale with 5.5 on the D scale, then read the answer 2.75 on the left index of the C scale. Although we have physically subtracted two quantities, we have in fact divided one number by another.

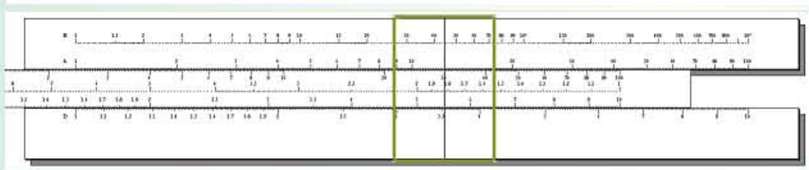
Now let us try a harder calculation: What is $2.1 \times 5.5 \times 7.8 / 32$?



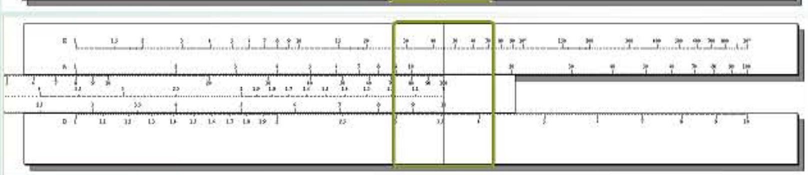
Move the cursor to 2.1 D.



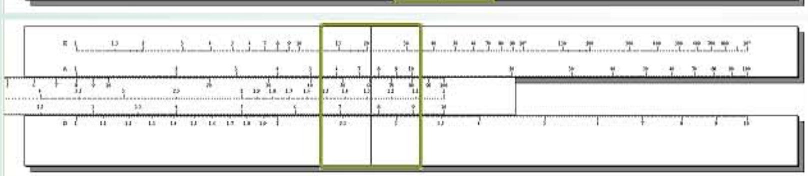
Line up 3.2 C with the cursor.



Move the cursor to 5.5 C.



Line up 10 C with the cursor. ($32 = 3.2 \cdot 10$)



Move the cursor to 7.8 C.

Read the answer on the D scale: 2.815.

Notice that we cannot use a slide rule for addition or subtraction. We can also calculate square and cube roots on a slide rule, although they require special scales. You can find out more about these scales and about slide rules in general on the Internet.

The pictures below show some different and unusual types of slide rule.

A circular slide rule

A watch with a powerful two-scale slide rule

A lighter with a working circular slide rule

CHAPTER 3 SUMMARY

1. Exponential Equations and Inequalities

- An exponential equation is an equation in which the unknown appears only in the exponent(s). For example, $3^{x+2} = 1$ is an exponential equation, but $x \cdot 3^x = 3$ is not an exponential equation. A solution of an exponential equation is a value of x which satisfies the equation. The set which contains all the solutions of an exponential equation is called the solution set of the equation. Two exponential equations are said to be equivalent if they have exactly the same solution set. Solving an exponential equation means finding its solution set.
- To solve an exponential equation, we try to reduce it to one of four main forms and then use the appropriate method.

Form	Solution
$a^{f(x)} = a^{g(x)}$	Solve $f(x) = g(x)$.
$a^{f(x)} = b$	• If $b \le 0$ then there is no solution. • Otherwise, solve $f(x) = \log_a b$.
$a^{f(x)} = b^{g(x)}$	Take logarithms (usually common logarithms) of both sides to a convenient base.
$f(x)^{g(x)} = f(x)^{h(x)}$	• Combine the solutions of $f(x) = 1$, $g(x) = h(x)$ and the system $\begin{cases} f(x) = 0 \\ g(x) > 0 \\ h(x) > 0 \end{cases}$. • Also check for solution(s) to $f(x) = -1$.

- To solve an exponential equation of the form $(p \cdot a^{2f(x)}) + (q \cdot b^{2f(x)}) + (r \cdot a^{f(x)} \cdot b^{f(x)}) = 0$, we divide both sides by any of the terms (excluding the coefficient) in order to obtain a known type of equation (quadratic, cubic, etc) which is easier to solve.
- To solve an exponential inequality of the form $a^{f(x)} > a^{g(x)}$, we solve

  $-f(x) > g(x)$ if $a > 1$,

  $-f(x) < g(x)$ if $a \in (0, 1)$.

2. Logarithmic Equations and Inequalities

- To solve a logarithmic equation, we try to reduce it to one of two main forms and then use the appropriate method.

Form	Solution
$\log_{f(x)} g(x) = b$	Solve $f(x)^b = g(x)$.
$\log_{f(x)} g(x) = \log_{f(x)} h(x)$	Solve $g(x) = h(x)$.

To complete the solution, we check the solutions against the existence conditions for logarithms in the original equation.

- We can solve a logarithmic inequality of the form $\log_{f(x)} g(x) < \log_{f(x)} h(x)$ by establishing and solving one of the systems

$$\begin{cases} f(x) > 1 \\ g(x) < h(x) \\ g(x) > 0 \end{cases} \quad \text{or} \quad \begin{cases} 0 < f(x) < 1 \\ g(x) > h(x) \\ h(x) > 0. \end{cases}$$

3. Systems of Equations and Inequalities

- To solve a system of equations or inequalities, we use the solution methods shown previously along with algebraic operations, elimination and substitution. To complete the solution, we check the solutions against the existence conditions for logarithms.
- We establish sign tables to solve systems of inequalities.

Concept Check

1. How do we solve exponential equations?
2. Why do we need to check the solutions of a logarithmic equation against the existence conditions for logarithms? Why do unsuitable solutions appear?
3. What is a critical value?
4. How do we establish a sign table? How do we know which sign to put in each interval?

CHAPTER REVIEW TEST 3A

1. Solve $\log_3(\log_8 x) = 0$ for x.

A) {3} B) {4} C) {5} D) {8} E) {16}

2. Solve $5^{2x+1} = \frac{1}{125}$ for x.

A) {3} B) {2} C) {1} D) {–1} E) {–2}

3. What is the solution set for $4^x - (9 \cdot 2^x) + 8 = 0$?

A) {0, 2} B) {0, 3} C) {2, 3}

D) {3, 4} E) {1, 8}

4. Solve $\log(2x + 1) - \log x = 2$.

A) $\frac{1}{104}$ B) $\frac{1}{102}$ C) $\frac{1}{100}$ D) $\frac{1}{99}$ E) $\frac{1}{98}$

5. What is the solution of the inequality $\log_3(4 - x) < 1$?

A) $x > 1$ B) $x > 4$ C) $x > -1$

D) $x \in (1, 4)$ E) $x \in (-1, 4)$

6. What is the sum of the solutions of

$\log_{27} a \cdot \log_9 a = \frac{1}{24}$?

A) $\sqrt{3}$ B) $\frac{4\sqrt{3}}{3}$ C) $\frac{5\sqrt{3}}{3}$ D) $2\sqrt{3}$ E) $\frac{7\sqrt{3}}{3}$

7. $\log_3 a^2 + \log_2 b^3 = 12$ and $\log_3 a^3 - \log_2 b^2 = 5$ are given. What is $\log_a(b - 1)$?

A) $\frac{1}{3}$ B) $\frac{1}{2}$ C) 1 D) 2 E) 3

8. Given $x \neq 1$ and $x^{\log_y x} = \sqrt{x}$, solve $\log_x y - \log_y x$ for x.

A) $\frac{3}{2}$ B) 2 C) $\frac{5}{2}$ D) 3 E) $\frac{7}{2}$

9. Solve $5^{x+1} = 2^x$ for x.

A) $\log_5 2$ B) $\log_2 5$ C) $\log_{\frac{5}{2}} 2$

D) $\log_{\frac{2}{5}} 2$ E) $\log_{\frac{2}{5}} 5$

10. What is the solution set for

$\log(3x - 4) - \log(3x + 1) = 0$?

A) $\{-\frac{1}{3}\}$ B) $\{\frac{4}{3}\}$ C) $\varnothing$

D) $\mathbb{R}$ E) $\mathbb{R} - \{-\frac{1}{3}, \frac{4}{3}\}$

11. What is the sum of the solutions of

$3^{\log x} + 3^{2 - \log x} = 10$?

A) $\frac{101}{100}$ B) 11 C) $\frac{1001}{10}$ D) 101 E) 1001

12. $9^x - (10 \cdot 3^{x+1}) + 3^4 = 0$ has solutions x_1 and x_2. What is $\log_{\sqrt{2}}(x_1 + x_2)$?

A) 8 B) 4 C) 2 D) $\frac{1}{2}$ E) $\frac{1}{4}$

13. What is the sum of the integers in the solution set for $\log_2(\log_4(x - 6)) < 0$?

A) 36 B) 33 C) 24 D) 19 E) 17

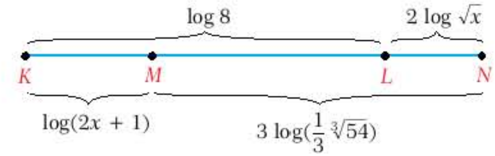
14. How many natural numbers satisfy the inequality $\log_3(3x - 5) \le \log_9(x + 6)^2$?

A) 3 B) 4 C) 5 D) 6 E) 7

15. What is $\log_{x+1}(x + 9)$ if x satisfies $\log_3(2x + 1) - \log_9(x - 2)^2 = 1$?

A) $\frac{4}{3}$ B) $\frac{5}{3}$ C) 2 D) $\frac{7}{3}$ E) 3

16.



In the figure above,

$KL = \log 8$, $LN = 2 \cdot \log \sqrt{x}$,

$KM = \log(2x + 1)$ and $MN = 3 \cdot \log(\frac{1}{3}\sqrt[3]{54})$.

What is x?

A) $\frac{1}{2}$ B) $\frac{1}{5}$ C) 1 D) 2 E) 5

17. If $\log_7(2x - 7) - \log_7(x - 2) = 0$, what is $\log_5 x$?

A) 0 B) 1 C) 2 D) 3 E) 4

18. Which of the following is the solution set for $\log_3(9 \cdot 3^{x+3}) = 3x + 1$?

A) $\{-1, 1\}$ B) $\{0, 2\}$ C) $\{0\}$

D) $\{1\}$ E) $\{2\}$

19. Solve $4^{\log_8 x^3} = 2x + 3$.

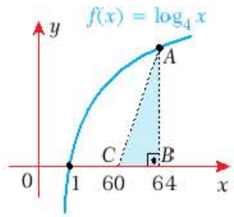
A) $\{-3\}$ B) $\{-1\}$ C) $\{1\}$ D) $\{2\}$ E) $\{3\}$

20. What is the product of the solutions of the equation $4^x + \frac{16}{4^x} = 10$?

A) $\frac{3}{4}$ B) $\frac{5}{4}$ C) 3 D) $\frac{15}{4}$ E) 3

CHAPTER REVIEW TEST 3B

1. What is the area of triangle ABC in the figure?



A) 32 B) 16 C) 6 D) 12 E) 24

2. If $\log_3(\log_7(\log_5 x)) = 0$, what is x?

A) 0 B) 1 C) 7^3 D) 7^5 E) 5^7

3. Solve $7^{1+\log_{49}(x+1)} = 35$.

A) {6} B) {12} C) {24} D) {48} E) {64}

4. Which value of x satisfies $5^x = 3^{x+1}$?

A) $\log_3 5$ B) $\log_5 3$ C) $\log_{\frac{3}{5}} 3$

D) $\log_{\frac{5}{3}} 3$ E) $\log_{15} 3$

5. Solve $\log_3(3 + 3\log_3 x) = 3$.

A) $\{3^2\}$ B) $\{3^4\}$ C) $\{3^5\}$ D) $\{3^6\}$ E) $\{3^8\}$

6. What is the solution set for the equation

$3\log_5 x + \log_x 5 = 4$?

A) $\{\sqrt[3]{5}, 5\}$ B) $\{\sqrt[3]{2}, 2\}$ C) $\{\sqrt{5}, 5\}$

D) $\{\sqrt{2}, 2\}$ E) $\{\sqrt[3]{3}, 3\}$

7. How many natural numbers satisfy the inequality $\log_5(x - 2) \le 2$?

A) 23 B) 24 C) 25 D) 26 E) 27

8. Which value of x satisfies

$2x - \log(5^{2x} + 4^x - 16) = x \cdot \log 4$?

A) 1 B) 2 C) 3 D) 4 E) 5

9. x_1 and x_2 are the two roots of the equation $(x^2 + 5)^{\log_3(x^2+5)} = 81$. Which of the following is a possible value of $x_1^{x_2}$?

A) 16 B) 9 C) $\frac{1}{4}$ D) $\frac{1}{9}$ E) $\frac{1}{27}$

10. What is the sum of all the solutions of $x^{\frac{1}{\log_x 5}} = 5$?

A) $\frac{11}{5}$ B) $\frac{126}{25}$ C) $\frac{26}{5}$ D) 6 E) 11

11. How many integers satisfy

$\log_3(x+1) - \log_{\frac{1}{3}}(3-x) \geq 1$?

A) 3 B) 4 C) 5 D) 6 E) 7

12. Which value of a satisfies

$\log_4 5 \cdot \log_5 6 \cdot \log_6 7 \ldots \log_{3a+1}(3a + 2) = \dfrac{5}{2}$?

A) 8 B) 10 C) 11 D) 12 E) 15

13. Which value of x satisfies $\log_{\frac{2}{5}}(x+3) \geq 1$?

A) 1 B) –2 C) $-\dfrac{14}{5}$ D) $-\dfrac{16}{5}$ E) $-\dfrac{46}{5}$

14. Solve $\log_{\sqrt[3]{2}} 4 = \log_{\sqrt{3}}(9^{\sqrt{x}})$.

A) $\{\dfrac{1}{9}\}$ B) $\{\dfrac{4}{9}\}$ C) $\{\dfrac{9}{4}\}$ D) $\{4\}$ E) $\{9\}$

15. What is the sum of the solutions of

$\log_6(x + 2) + \log_6(x + 7) = 2$?

A) 2 B) 3 C) 4 D) 5 E) 6

16. What is the solution set for $\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{1}{2}$?

A) $\{\frac{1}{2}\ln 3, -\frac{1}{2}\ln 3\}$ B) $\{\ln \sqrt{3}\}$ C) $\{2 \ln 3\}$

D) $\{\frac{1}{2}\ln \sqrt{3}\}$ E) $\{-\ln \sqrt{3}\}$

17. What is the product of the solutions of

$\log_3(10 - 3^x) = 10^{\log(2-x)}$?

A) –2 B) –1 C) 0 D) 1 E) 2

18. What is the solution set for the equation

$\log_3 x \cdot \log_x 5 = \log_3 5$?

A) $\{3, 5\}$ B) $\{3, 5, 15\}$ C) $\mathbb{R}$

D) $\mathbb{R} - \{1\}$ E) $\mathbb{R}^+ - \{1\}$

19. Which value cannot be taken by x if

$(x-1)^{\log_2(16-x^2)} = 1$?

A) $-\sqrt{15}$ B) $\sqrt{15}$ C) 0 D) 2 E) 4

20. What is the solution set for $\log_2(x^2 - 1) \leq 3$?

A) $(1, 3]$ B) $[-3, -1)$ C) $[-3, 3]$

D) $(-1, 1)$ E) $[-3, -1) \cup (1, 3]$

CHAPTER REVIEW TEST 3C

1. Solve $\log_2(\log_3 x) = 4$ for x.

A) $\{3^8\}$ B) $\{3^{12}\}$ C) $\{3^{16}\}$ D) $\{2^7\}$ E) $\{2^{12}\}$

2. Solve $3^{\log_9 x} = 3$ for x.

A) $\{9\}$ B) $\{3\}$ C) $\{\sqrt{3}\}$ D) $\{\sqrt[3]{3}\}$ E) $\{\sqrt[4]{3}\}$

3. Which ordered pair satisfies the system $\begin{cases} 5^{y+1} - 3^{x+1} = 116 \\ 5^y + 3^x = 28 \end{cases}$?

A) $(-2, 1)$ B) $(1, 2)$ C) $(2, 1)$ D) $(-1, 2)$ E) $(2, -1)$

4. Solve $10^{\ln x} + (3 \cdot x^{\ln 10}) = 40$ for x.

A) $\{e^2\}$ B) $\{e\}$ C) $\{\frac{1}{e}\}$ D) $\{\frac{1}{e^2}\}$ E) $\{\frac{1}{e^4}\}$

5. What is the sum of the solutions to the equation $3\sqrt{\log_2 x} - \log_2 8x + 1 = 0$?

A) 2 B) 8 C) 14 D) 16 E) 18

6. How many integers satisfy the inequality $\log_2(4x - 2) \leq 3$?

A) 1 B) 2 C) 3 D) 4 E) 5

7. What is the product of the values of x which satisfy $2 \log_2 x + \log_x 2 = 3$?

A) $\frac{1}{2}$ B) $\sqrt{2}$ C) 2 D) $2\sqrt{2}$ E) 4

8. Solve $2^{\ln a} - (3 \cdot a^{\ln 2}) = -8$ for a.

A) $\{\frac{1}{e^2}\}$ B) $\{\frac{1}{e}\}$ C) $\{1\}$ D) $\{e\}$ E) $\{e^2\}$

9. Find the sum of the solutions to the equation $\sqrt{\log_2 x} = \log_2 \sqrt{x}$.

A) 3 B) 5 C) 9 D) 17 E) 33

10. Solve $\log_3(x - 4) + \log_9(x^2 + 8x + 16) = 2$.

A) $\{5\}$ B) $\{6\}$ C) $\{7\}$ D) $\{9\}$ E) $\{16\}$

11. Which of the following is the solution set for $\log_3(3-x) \le 2$?

A) $(-9, 6]$ B) $[-6, 3)$ C) $(-6, -3)$

D) $[-3, 6)$ E) $(-9, 3)$

12. How many integer solutions does $(\log_5 x)^2 - 4 \le 0$ have?

A) 25 B) 26 C) 27 D) 28 E) 29

13. What is the solution set for

$\log_x 2 + \log_x 4 + \log_x 8 + \ldots + \log_x 256 = 12$?

A) $\{\frac{1}{2}\}$ B) $\{2\}$ C) $\{4\}$ D) $\{8\}$ E) $\{12\}$

14. Which value of x satisfies the equation

$\ln(x+1) - \ln x = -1 + \ln 3$?

A) $\dfrac{3}{e+3}$ B) $\dfrac{3}{e-3}$ C) $\dfrac{e}{3-e}$

D) $\dfrac{e}{e+3}$ E) $\dfrac{3e}{3-e}$

15. What is the solution set for $\log_x(2x-4) \le 1$?

A) $(0, 2]$ B) $(2, 4]$ C) $[\frac{1}{2}, 2)$

D) $(\frac{1}{2}, 4]$ E) $(3, 5]$

16. $(13 \cdot 6^x) - (6 \cdot 4^x) - (6 \cdot 9^x) = 0$ has solutions x_1 and x_2. Find $5^{x_1 \cdot x_2}$.

A) 25 B) 5 C) 1 D) $\dfrac{1}{5}$ E) $\dfrac{1}{25}$

17. Solve $\log_3(7 - \log_2(\frac{x}{3})) = 2$.

A) $\{\frac{1}{4}\}$ B) $\{\frac{1}{3}\}$ C) $\{\frac{2}{3}\}$ D) $\{\frac{3}{4}\}$ E) $\{3\}$

18. Which value of a satisfies $(\sqrt[a]{a})^{\log_{\sqrt{a}} 9} = 27$?

A) $\dfrac{4}{3}$ B) $\dfrac{3}{2}$ C) $\dfrac{5}{3}$ D) 2 E) $\dfrac{5}{2}$

19. Which value of x satisfies $\log x - 5\log 3 = -2$?

A) 1.25 B) 0.81 C) 2.43

D) 0.8 E) 0.8 or 1.25

20. Which values of x satisfy $2 \le \log(2x - 100) < 3$?

A) $50 \le x < 250$ B) $100 \le x < 120$

C) $250 \le x < 500$ D) $300 \le x < 550$

E) $100 \le x < 550$

1. x and y are integers such that $(x \cdot \log_{400} 5) + (y \cdot \log_{400} 2) = 3$. Calculate $x + y$.

A) 13 B) 15 C) 18 D) 20 E) 21

2. Which value of x satisfies $2^{\log_5 x^2} - 2^{1+\log_5 x} + 2^{-1+\log_5 x} = 1$?

A) 25 B) 5 C) 1 D) $\frac{1}{5}$ E) $\frac{1}{25}$

3. In the triangle opposite,

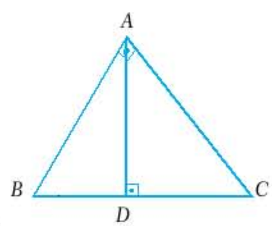
$|AD| \perp |BC|$, $|AB| \perp |AC|$,

$AB = \log y$,

$AC = \log 3$,

$AD = \log x$ and $BC = \log 9$.

What is $\frac{y}{x}$?



A) x^2 B) x C) 1 D) $\frac{1}{x}$ E) $\frac{1}{x^2}$

4. What is the sum of the solutions to $(\ln x)^2 = \ln x^2$?

A) $\frac{1}{e^2}+1$ B) $e - 1$ C) $\frac{1}{e}+1$ D) $e^2 + 1$ E) $e(e + 1)$

5. Solve $x^{\log_6 3} + 3^{\log_6 x} = 18$.

A) $\{3\}$ B) $\{6\}$ C) $\{18\}$ D) $\{36\}$ E) $\{54\}$

6. How many integer values of x satisfy the inequality $-1 < \log_2(\log_4(x - 2)) < 1$?

A) 16 B) 15 C) 13 D) 11 E) 10

7. What is the sum of the values of x and y which satisfy the system $\begin{cases} \log_3 x^5 - \log_5 y^3 = 7 \\ \log_{\sqrt[3]{3}} x + \log_{\sqrt{5}} y = 8 \end{cases}$?

A) 15 B) 14 C) 13 D) 10 E) 9

8. What is the solution set for $1 + \log_x 3 > 2 \log_x 4$?

A) $(-\infty, \frac{16}{3})$ B) $(1, \frac{16}{3})$ C) $(\frac{16}{3}, \infty)$ D) $(0, 1) \cup (\frac{16}{3}, \infty)$ E) $(0, 1)$

9. $x \neq 1$ and $\log_{xy} \frac{y}{x} + \log_{y^2} x = 1$ are given. What is $\log_x y$?

A) $-\frac{1}{2}$ B) -2 C) $\frac{1}{2}$ D) 2 E) 4

10. How many different integers satisfy the inequality $\log(x^2 + 21) < 1 + \log x$?

A) 3 B) 4 C) 5 D) 6 E) 7

11. What is the product of the solutions to

$\log^2(100x) + \log^2(10x) = 14 + \log\dfrac{1}{x}$?

A) $10^{-1/2}$ B) $10^{-3/2}$ C) $10^{-5/2}$

D) $10^{-7/2}$ E) $10^{-9/2}$

12. Solve $(3\log_5 2) + 2 - x = \log_5(3^x - 5^{2-x})$.

A) {6} B) {5} C) {4} D) {3} E) {2}

13. Which of the following is the solution set for $x^{\ln x} \le e$?

A) $[\frac{1}{e}, e]$ B) $[\frac{1}{e^2}, e]$ C) $[1, 3]$

D) $[1, e^2]$ E) $[\frac{1}{e}, 1]$

14. Given $\log x + \log_{100} x = 7$, calculate $x^{3/2}$.

A) $10^{14/3}$ B) 10^7 C) 10^6 D) 10^5 E) 10^4

15. What is the product of the solutions to $(\ln x)^2 - \ln x^5 - \ln e^6 = 0$?

A) e^6 B) e^5 C) e^{-6} D) e^{-5} E) e^{-1}

16. Solve $(2x)^{\log_b 2} - (3x)^{\log_b 3} = 0$ for $x > 0$ and $b > 0$.

A) $\{\frac{1}{216}\}$ B) $\{\frac{1}{6}\}$ C) {1} D) {6} E) {12}

17. Which value of x satisfies $2^{2x} - (8 \cdot 2^x) + 12 = 0$?

A) $\log 3$ B) $\frac{1}{2}\log 6$ C) $1 + \log\frac{3}{2}$

D) $1 + \log_2 3$ E) $\dfrac{\log 3}{\log 2}$

18. What is the solution set for $x^{\log x} = \dfrac{x^3}{100}$?

A) $\{\frac{1}{10}\}$ B) {10} C) {100}

D) {10, 100} E) $\{\frac{1}{10}, 10\}$

19. Find the sum of the solutions to $x^{3-\log_6 x} = 36$.

A) 42 B) 41 C) 40 D) 36 E) 70

20. What is the solution set for the inequality $\log_5(x+1) + \log_5(x-1) \le \log_5 3$?

A) $-2 \le x \le 2$ B) $-2 < x < 2$ C) $-1 < x \le 2$

D) $1 \le x \le 2$ E) $1 < x \le 2$

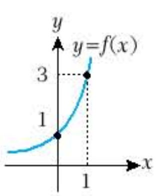
ANSWERS TO EXERCISES

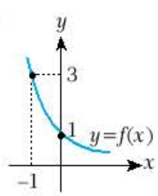
EXERCISES 1.1

1. a. $\frac{1}{9}$ b. $\frac{1}{9}$ c. 2 d. $\frac{4}{9}$ e. 1 f. 3 g. $\frac{9}{4}$ h. $\frac{1}{729}$ i. $\frac{5}{6}$ j. $\frac{36}{13}$ k. $\frac{7}{8}$ l. -3^{19} m. 3^{47} 2. a. a^8 b. $\frac{1}{a^8}$ c. x^8 d. $\frac{1}{x^6}$ e. $\frac{p^3}{2}$ f. $\frac{x^6}{16}$ g. $9x^2$ h. $3x^4y^2$ i. x^2 j. $\frac{1}{x^3}$ k. x^{11} l. $\frac{1}{x^{10}}$ 3. a. 3 b. 3 c. 2 4. a. $3\cdot\frac{x^3}{y^2}$ b. $x\sqrt[12]{x}$ c. $x\sqrt{3x}$ d. $\frac{1}{\sqrt{2}x^4}$ e. $\sqrt{xy}$ f. $-\frac{x}{2y^5}$ 5. a. 3 b. $\frac{1}{3}$ c. 8 d. $\frac{1}{8}$ e. -5 f. not a real number g. 2 h. $\frac{1}{2}$ i. -2 j. $\frac{5}{2}$ k. $-\frac{3}{2}$ l. $\frac{1}{6}$ m. $\sqrt{2}$ n. 4 o. 9 6. a. $\frac{1}{3a}$ b. $\frac{2}{m^2}$ c. a^2 d. $\frac{3}{n^2}$ e. $8x$ f. $\frac{12}{x}$ 7. a. 1 b. $25^{\sqrt{3}}$ c. $\frac{1}{64}$ d. $\frac{1}{2}$ e. $\frac{1}{4^{\sqrt{3}}}$ f. $\frac{1}{36}$ 8. a. a^4 b. m 9. a. 7 b. 18

EXERCISES 1.2

1. a. yes b. yes c. yes d. yes e. no f. no g. no

2. a.


b.


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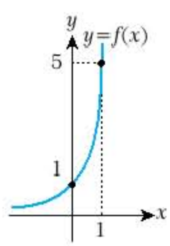

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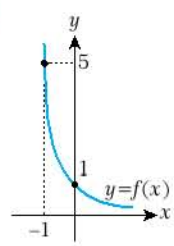

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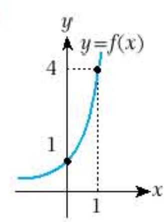

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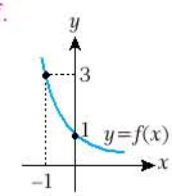


3. a. $f(x) = 3\cdot 2^x$ b. $f(x) = 3\cdot(\frac{1}{2})^x$ c. $f(x) = 3^x$ 4. a. $4^{\frac{7}{8}}$ b. $(\frac{4}{5})^{10}$ c. $(0.7)^{\frac{2}{3}}$ d. $(\frac{1}{\sqrt{6}})^{0.6}$ e. $\sqrt[5]{(\sqrt{2}-1)^{12}}$ f. y

5. a. no b. yes c. no d. yes e. no f. yes 6. a. ↗ b. ↘ c. ↗ d. ↘ e. ↗ f. ↗

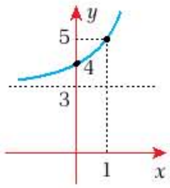
EXERCISES 1.3

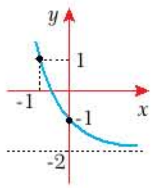
1. a. shift 1 unit up b. shift 3 units down c. shift 2 units right d. shift 1 unit left e. reflect in the x-axis f. reflect in the y-axis g. stretch horizontally by a factor of 2 h. stretch vertically by a factor of 3 i. shift 2 units right, reflect in the x-axis j. shift 1 unit left, reflect in the x-axis, shift 1 unit up

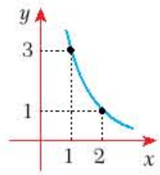
2. a. $y=(\frac{1}{2})^{-x}$ b. $y=(\frac{1}{2})^{x-2}$ c. $y=(\frac{1}{2})^{x}-3$ d. $y=3\cdot(\frac{1}{2})^{x}$

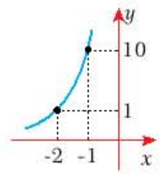
3. a. D: $\mathbb{R}$, R: $\mathbb{R}^+$, a: $y=0$ b. D: $\mathbb{R}$, R: $\mathbb{R}^+$, a: $y=0$ c. D: $\mathbb{R}$, R: $\mathbb{R}^+$, a: $y=0$ d. D: $\mathbb{R}$, R: $\mathbb{R}^-$, a: $y=0$ e. D: $\mathbb{R}$, R: $(-3,\infty)$, a: $y=-3$ f. D: $\mathbb{R}$, R: $(4,\infty)$, a: $y=4$ g. D: $\mathbb{R}$, R: $(-\infty,6)$, a: $y=6$ h. D: $\mathbb{R}$, R: $\mathbb{R}^-$, a: $y=0$

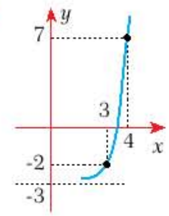
i.. D: $\mathbb{R}$, R: $\mathbb{R}^-$, a: $y=0$

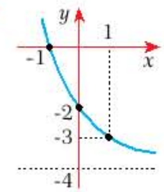
4. a.


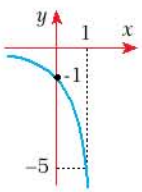
b.


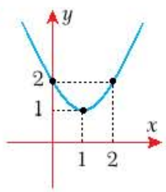
c.


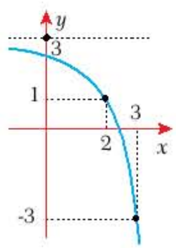
d.


e.


f.


g.


h.


i.


EXERCISES 1.4

1. a. 50 b. 70 c. 400 2. a. 832 918 b. 142 7476 3. 100 minutes 4. 638 976 5. a. \$910 b. \$828.10 c. \$322.48 6. 612 days 7. 24.8 years 8. 55.2 days 9. a. \$1220.39 b. \$1489.35 c. \$1817.59 10. \$473.65

EXERCISES 2.1

1. a. 1 b. –3 c. 1 d. 0 e. 2 f. $\frac{3}{5}$ g. 0 h. 0 i. $\frac{1}{3}$ j. 3 k. 1 l. 0 2. a. $\log_3 4$ b. $\log_2\frac{3}{2}$ c. $\log_3\frac{9}{4}$ d. ln 16 e. log 5 f. log 20

3. a. x b. 3 c. 49 d. $\frac{\sqrt{2}}{2}$ e. $\frac{1}{64}$ f. 5 g. $\frac{10}{3}$ h. 3 i. $\frac{\sqrt[3]{3}}{3}$ 4. a. 1.2552 b. 1.4771 c. –0.699 5. a. 16 b. 10 c. 13 d. 26 6. a. $\log\frac{\sqrt[3]{x}\cdot z^2}{y}$ b. $\log\sqrt{\frac{yz}{x}}$ 7. a. $3\log a + 2\log b + \log c$ b. $\frac{1}{3}\log a+\frac{1}{2}\log b+\log c$ 8. a. 1 b. 3 c. $\frac{3}{2}$ d. 3

9. a. $\frac{1}{a}$ b. $2(1-a)$ c. $\frac{p+1}{p}$ d. $\frac{a+3}{2}$ e. $\frac{2ab+2a-1}{ab+b+1}$ 10. a. $\frac{3a+2}{3+2a}$ b. $\frac{3+4a}{1-2a}$ 11. a. $-\frac{1}{2}$ b. $\frac{1}{2}$ 12. a. $\frac{1000}{9}$ b. $\frac{1125}{16}$

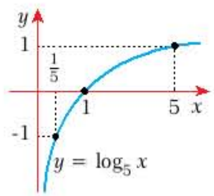
EXERCISES 2.2

1. a. $(2, \infty)$ b. $\mathbb{R} \setminus [-1, 6]$ c. $\mathbb{R} \setminus \{1\}$ d. $\mathbb{R} \setminus \{0\}$ e. $(-1, 1)$ f. $(-\infty, 0) \cup (2, 3)$ g. $\mathbb{R} \setminus \{1\}$ h. $(-3, 0) \cup (1, \infty) \setminus \{-2\}$

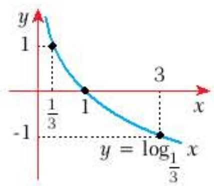
2. a. $(2, \infty)$ b. $(1, \infty)$ c. $(0, 1)$ d. $(\frac{1}{2}, \infty) \setminus \{1\}$ e. $(-2, 1)$ f. $\mathbb{R} \setminus [-2, 2]$ g. $\mathbb{R} \setminus [0, 1]$ h. $(\frac{4}{3}, 2)$ i. $(-2, -\frac{3}{2})$

j. $\mathbb{R} \setminus [-2, 1]$ k. $(2, 3)$ l. $(-\infty, 0)$

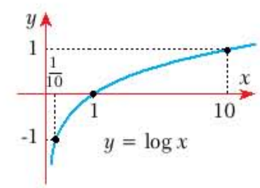
3. a.



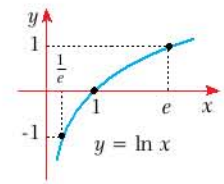
b.



c.



d.



4. a. positive for $x > 1$, negative for $\frac{1}{2} < x < 1$ b. positive for $x > 2$ or $x < 0$, negative for $0 < x < 2$ 5. a. $<$

b. $>$ c. $>$ d. $<$ 6. a. $x < y < z$ b. $z < y < x$ c. $z < y < x$ 7. a. $f^{-1}(x) = \log_5 x$ b. $f^{-1}(x) = \log_{\frac{2}{3}} x$ c. $f^{-1}(x) = \ln x$

d. $f^{-1}(x) = \log_{\frac{1}{5}} x$ e. $f^{-1}(x) = 1 + \log_2 x$ f. $f^{-1}(x) = \dfrac{1 + \log_3(x-1)}{2}$ g. $f^{-1}(x) = -1 + \ln(2 - x)$ h. $f^{-1}(x) = 2^x + 3$

i. $f^{-1}(x) = e^x - 2$ j. $f^{-1}(x) = 10^{\frac{x}{3}} - 1$ k. $f^{-1}(x) = 10^{x-1}$ l. $f^{-1}(x) = 3^x + 4$ 8. a. strictly decreasing b. strictly decreasing

9. $m \in \{-3, 3\}$ 10. a. $a < b$ b. $a > b$ 11. 2 12. 73

EXERCISES 2.3

1. 5.08 2. the San Francisco earthquake was twice as intense as the Indonesian earthquake 3. 1.5 4. 15 times
5. a. 2 b. 4 c. 5 6. 7 7. $3.98 \cdot 10^{-3}$ mol/L 8. between 10^{-1} and 10^{-14} mol/L 9. 70 dB 10. 27 dB 11. 22 dB

EXERCISES 3.1

1. a. $\{\frac{3}{2}\}$ b. $\{-\frac{5}{3}\}$ c. $\{-1, 2\}$ d. $\{\frac{2}{3}\}$ e. $\{2\}$ f. $\{4\}$ g. $\{2\}$ h. $\{\frac{1}{2}, 3\}$ 2. a. $\{\log_2 10\}$ b. $\{0\}$ c. $\{\pm\sqrt{\log_5 4}\}$

d. $\{\log_3 \sqrt{6}\}$ e. $\{\dfrac{-2 + \log_2 3}{3}\}$ f. $\{\log_3 2\}$ 3. a. $\{\log_{30} \dfrac{175}{3}\}$ b. $\{0\}$ c. $\{\log_{\frac{64}{125}} 40\}$ d. $\{0\}$ e. $\{2\}$

4. a. $\{2\}$ b. $\{-2\}$ c. $\{\frac{3}{2}\}$ d. $\{\log_3 2, \log_3 7\}$ e. $\{\log_{1+\sqrt{3}} 2\}$ 5. a. $\{\ln 4\}$ b. $\{4\}$ c. $\{1\}$ d. $\{\log_{\frac{15}{28}} 98\}$

6. a. $\{-2\}$ b. $\{-1, 1\}$ c. $\{-1, 0, 1\}$ d. $\{0, \log_{\frac{5}{2}} 2\}$ e. $\{0, \log_{\frac{2}{5}} 2\}$ 7. a. $\{0\}$ b. $\{\log_{\frac{2}{3}} 2, 0\}$ c. $\{1, \log_3 2\}$

8. a. $\{3\}$ b. $\{3\}$ 9. a. $\{1, 4\}$ b. $\{-1, 1, 2\}$ 10. a. $(5, \infty)$ b. $(-\infty, \frac{1}{4})$ c. $[3, \infty)$ d. $[4, \infty)$ e. $\mathbb{R}$

11. a. $(0, 1)$ b. $(2, \infty)$ c. $(-\infty, 1)$ 12. a. $[-1, \infty)$ b. $(-\infty, -1]$ c. $\{0, \frac{1}{2}\}$ 13. a. $(-\infty, 0) \cup (\log_2 3, \infty)$ b. $(-\infty, 0) \cup [1, \infty)$

14. $(-\infty, -\frac{1}{2}) \cup (1, \infty)$

EXERCISES 3.2

1. a. $\{4\}$ b. $\{3\}$ c. $\varnothing$ d. $\{5\}$ e. $\{-1, \frac{7}{2}\}$ f. $\{256\}$ g. $\{41\}$ 2. a. $\{2\}$ b. $\{6\}$ c. $\{2\}$ d. $\{1\}$ 3. a. $\{1\}$ b. $\{3^{-\frac{5}{3}}, 3\}$ c. $\{2\}$ d. $\{2\}$ e. $\{1, e\}$ f. $\{e\}$ g. $\{\frac{1}{2}\}$ h. $\{1\}$ 4. a. $\{16\}$ b. $\{2\}$ c. $\{4\}$ d. $\{\frac{1}{9}, 9\}$

5. a. $(\frac{5}{2}, \frac{9}{2})$ b. $[-1, \infty)$ c. $(2, \frac{72}{32})$ d. $[-5, -3) \cup (3, 5]$ e. $(-2, -\sqrt{3}) \cup (\sqrt{3}, 2)$ f. $(0, 1]$ g. $(-2, 1)$ h. $(1, 16)$ i. $(4, 13)$ j. $(7, 11]$ k. $(3, 5)$ 6. $(0, a) \cup (1, \frac{1}{a})$

EXERCISES 3.3

1. a. $\{(3, \frac{1}{2})\}$ b. $\{(2, 1), (1, 2)\}$ c. $\{(2, \frac{3}{2})\}$ d. $\{(-\frac{5}{2}, -\frac{3}{2})\}$ e. $\{(1, 0)\}$ f. $\{(2, 1)\}$ 2. a. $\{(\frac{1}{6}, 1)\}$ b. $\{(0, -2), (2, 0)\}$ c. $\{(-2, 0)\}$ d. $\{(1, 2)\}$ e. $\{(\frac{3}{5}, \frac{1}{5})\}$ f. $\{(2, -\frac{1}{2})\}$ 3. a. $\{(2\sqrt{2}, \sqrt[4]{8})\}$ b. $\{(5, 2)\}$ c. $\{(2, 6)\}$ d. $\{(1, 4), (4, 1)\}$ e. $\{(3, 9), (9, 3)\}$ 4. a. $\{(8, 2), (2^{-\frac{8}{3}}, 2^{\frac{20}{3}})$ b. $\{(27, 4)\}$ c. $\{(4, 16), (16, 4)\}$ d. $\{(5, 5)\}$ e. $\{(2, 4), (4, 2)\}$ 5. a. $(-1, 3)$ b. $(\frac{3}{2}, 2) \cup (3, \infty)$ c. $[2 + 2\sqrt{3}, \infty)$ d. $(-\infty, -2]$ e. $(2, 3]$

ANSWERS TO TESTS

TEST 1A

1.	D	11.	B
2.	A	12.	A
3.	E	13.	D
4.	E	14.	D
5.	E	15.	D
6.	C	16.	E
7.	B	17.	C
8.	D	18.	D
9.	B	19.	E
10.	B	20.	D

TEST 1B

1.	A	11.	B
2.	D	12.	C
3.	D	13.	E
4.	D	14.	C
5.	D	15.	D
6.	C	16.	D
7.	C	17.	C
8.	D	18.	A
9.	D	19.	B
10.	E	20.	C

TEST 2A

1.	A	11.	B
2.	D	12.	C
3.	C	13.	E
4.	B	14.	D
5.	D	15.	B
6.	A	16.	C
7.	B	17.	B
8.	B	18.	B
9.	D	19.	B
10.	D	20.	D

TEST 2B

1.	C	11.	B
2.	A	12.	C
3.	A	13.	D
4.	E	14.	C
5.	C	15.	E
6.	A	16.	C
7.	B	17.	A
8.	D	18.	D
9.	E	19.	D
10.	C	20.	C

TEST 2C

1.	C	11.	B
2.	C	12.	C
3.	B	13.	B
4.	A	14.	A
5.	E	15.	C
6.	B	16.	A
7.	D	17.	D
8.	D	18.	D
9.	B	19.	D
10.	B	20.	A

TEST 2D

1.	E	11.	B
2.	C	12.	A
3.	E	13.	E
4.	B	14.	C
5.	D	15.	B
6.	A	16.	A
7.	E	17.	D
8.	C	18.	C
9.	D	19.	C
10.	D	20.	C

TEST 3A

1.	D	11.	D
2.	E	12.	B
3.	B	13.	E
4.	E	14.	B
5.	D	15.	A
6.	B	16.	A
7.	A	17.	B
8.	A	18.	E
9.	E	19.	E
10.	C	20.	A

TEST 3B

1.	C	11.	A
2.	E	12.	B
3.	C	13.	C
4.	D	14.	C
5.	E	15.	A
6.	A	16.	B
7.	C	17.	C
8.	B	18.	E
9.	C	19.	E
10.	C	20.	E

TEST 3C

1.	C	11.	B
2.	A	12.	A
3.	B	13.	D
4.	B	14.	C
5.	E	15.	B
6.	B	16.	D
7.	D	17.	D
8.	E	18.	D
9.	D	19.	C
10.	A	20.	E

TEST 3D

1.	C	11.	D
2.	B	12.	E
3.	B	13.	A
4.	D	14.	B
5.	D	15.	B
6.	C	16.	B
7.	B	17.	D
8.	D	18.	D
9.	A	19.	A
10.	A	20.	E

GLOSSARY

A

antilogarithm: The antilogarithm of a number a is the number whose logarithm is equal to a. x is the antilogarithm of $\log x$: $x = \text{antilog}(\log x)$ or $x = \log^{-1}(\log x)$.

argument: 1. a variable in a mathematical expression whose value determines the dependent variable (such as x in $f(x) = y$). 2. the number or expression denoted by x in the logarithmic expression $\log_a x$.

asymptote: a line which the graph of a function approaches but never touches.

B

base: 1. a number (such as 5 in 5^7) that is raised to a power. 2. the number or expression denoted by a in the logarithmic expression $\log_a x$.

basic exponential function: a function of the form $f(x) = a^x$ for $a > 0$, $a \neq 1$.

basic logarithmic function: a function of the form $f:(0, \infty) \to \mathbb{R}$, $f(x) = \log_a x$ for $a > 0$, $a \neq 1$.

bijection (bijective function): a mathematical function that is both one-to-one and onto.

C

characteristic: the integer part of a common logarithm. 2 is the characteristic of $\log 500 = 2.699$.

cologarithm: the logarithm of the reciprocal of a positive number x, written $\text{colog}\, x$.

common logarithm: a logarithm to the base 10. The common logarithm of a number x is written as $\log x$.

compound interest: interest which is calculated based on the sum of an original principal and any previous interest.

critical value: a value of a variable which makes all or part of an expression zero.

cube root: The cube root of a number x is the number whose cube is x. 2 is the cube root of 8 since $2^3 = 8$.

D

decibel scale: a scale based on common logarithms, which is used to measure the relative intensity of sounds.

decreasing function: a function whose value never increases as the value of the variable increases.

E

e (Euler number): the irrational number 2.71828183..., which forms the base of natural logarithms.

equivalent: If two equations have the same solution set then they are called equivalent equations.

Euler logarithm: another name for a *natural logarithm*.

exponent: a number written above and to the right of an expression, which shows the power to which the expression should be raised. 2 is the exponent in both 3^2 and $(x - 5)^2$.

exponential decay: a change in the amount of a quantity over time, which can be modeled by a decreasing exponential function.

exponential equation: an equation in which the variable appears in an exponent.

exponential function: a function whose independent variable is in the exponent, such as $f(x) = a^x$ for $a > 0, a \neq 1$ or $f(x) = c \cdot a^{d(x+p)} + k$ for $a, c, d, p, k \in \mathbb{R}$ and $a > 0$, $a \neq 1$.

exponential growth: a change in the amount of a quantity over time, which can be modeled by an increasing exponential function.

H

half-life: the time required for half of the atoms in a radioactive substance to decay.

horizontal shift: moving a graph to the left or right without changing its shape.

horizontal shrink: changing the shape of a graph to fit a smaller horizontal scale.

horizontal stretch: changing the shape of a graph to a fit a bigger horizontal scale.

I

identity function: the function defined by $f(x) = x$.

increasing function: a function whose value never decreases as the value of the variable increases.

index (plural indices): 1. another word for *exponent*. 2. the number in a radical expression which shows the root to be extracted. 3 is the index in $\sqrt[3]{64}$.

injection (injective function): a mathematical function that is a one-to-one mapping.

integer: any number which is a member of the set $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

interest: a charge for borrowed money, or the profit in goods or money that is made on invested capital, calculated as a percentage of the borrowed money or capital.

interpolation: a way of estimating the value of a function between two points by assuming that the graph of the function is a straight line.

irrational number: any real number which cannot be expressed as a fraction.

L

logarithm: The logarithm x of a number N to a base a is the power to which a must be raised in order to obtain N: $x = \log_a N$ means $a^x = N$.

logarithmic function: a function whose independent variable is in a logarithm, such as $f(x) = \log_a x$. The inverse of a logarithmic function is an exponential function.

logarithmic spiral (equiangular spiral): a spiral whose radius grows exponentially with its angle to the x-axis. It is defined by the polar equation $r = a \cdot e^{b\theta}$ where r is the distance from the origin, θ is the angle with the x-axis, and a and b are arbitrary constants.

M

mantissa: the part of a logarithm which lies to the right of the decimal point. 0.699 is the mantissa of $\log 500 = 2.699$.

monotone (monotonic) function: a function which is either increasing or decreasing.

N

natural logarithm: a logarithm to the base e. The natural logarithm of a number x is written $\ln x$.

natural number: any number which is a member of the set $\{1, 2, 3, \ldots\}$.

nth root: b is an nth root of a if and only if $b^n = a$.

O

one-to-one function: a function for which $f(x_1) \neq f(x_2)$ for any $x_1 \neq x_2$.

onto function: a function for which, for any y in the range, there is at least one x in the domain such that $f(x) = y$.

P

parameter: a variable that can be varied or changed in an expression.

pH scale: a scale which is used to measure the amount of acid in a solution. A low pH number represent a high acidity, and a high pH number represents a high basicity.

power: a number or expression which is the result of a number multiplying a number by itself, as indicated by an exponent. 8 is the third power of 2: $2^3 = 8$.

principal: a sum of money which is borrowed or invested.

principal nth root: the positive nth root of a number. 2 and –2 are square roots of 4, but only 2 is the principal square root of 4.

R

radical sign: the sign $\sqrt{\ }$ or $\sqrt[n]{\ }$ (where n is an integer greater than or equal to 2) in a radical expression.

radicand: the quantity under a radical sign. 27 is the radicand in $\sqrt[3]{27}$.

rational number: any real number which can be expressed as a fraction.

real number: any rational or irrational number.

reflection: changing the shape of a graph by reflecting it along a line.

Richter scale: a scale based on common logarithms which is used to describe the strength of an earthquake.

root of an equation (solution): a value of a variable in an equation which makes the equation true.

root of a number: a number which equals a given number when raised to an integer power. 3 and –3 are roots of 9.

S

scientific notation: a number in scientific notation is expressed as the product of a number between 1 and 10 and an appropriate power of 10. 1.591×10^3 is a number in scientific notation. 1591 is the same number in normal notation.

slide rule: a device used for multiplying and dividing numbers which makes use of the property $\log(ab) = \log a + \log b$. A slide rule has a body and a slide marked with logarithmic scales, and a movable cursor.

square root: A square root of a number x is a number whose square is x, denoted by $\sqrt{x}$. The square root of 9 is 3: $\sqrt{9} = 3$.

surjection (surjective function): a mathematical function that is an onto function.

T

transcendental number: a number that is real but not algebraic, i.e. it is not a root of any polynomial equation with rational coefficients. e and π are examples of transcendental numbers.

V

vertical shift: moving a graph upwards or downwards without changing its shape.

vertical shrink: changing the shape of a graph to fit a smaller vertical scale.

vertical stretch: changing the shape of a graph to a bigger vertical scale.

WHAT IS A LOGARITHM?